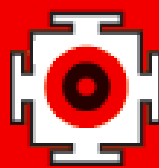
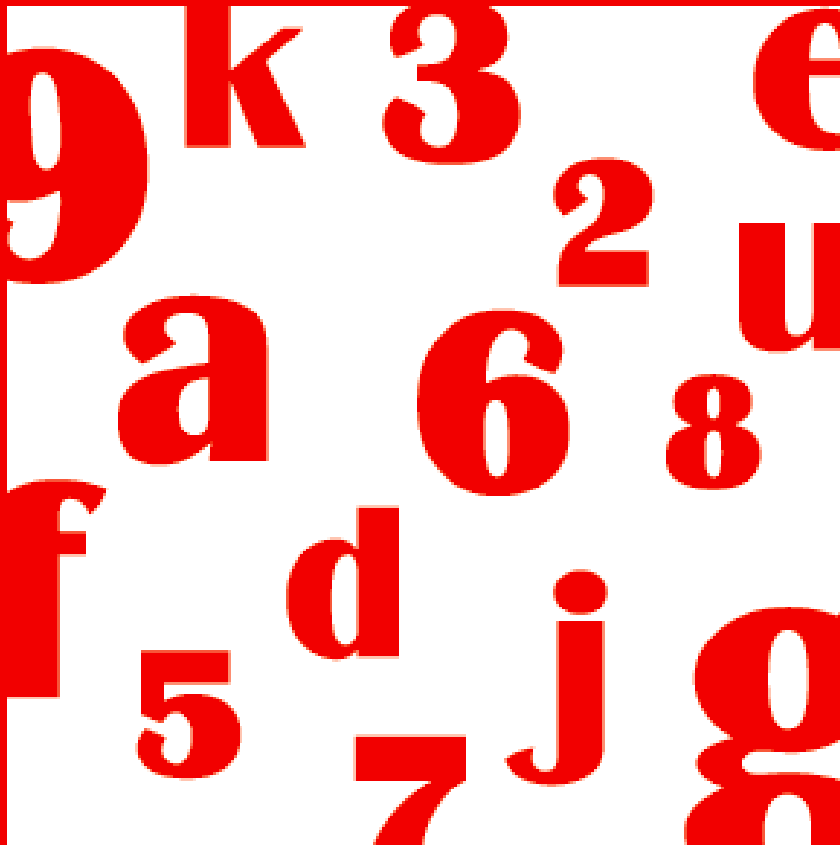


MISCELLANEOUS QUANT TOPICS



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www.takshzila.com

MRP: Rs. 350

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Set Theory

A set is any collection of objects satisfying a particular condition.

E.g. set of people drinking tea or set of people drinking coffee, set of students playing cricket, set of households subscribing to Times of India, etc. These are the sets that are going to be most commonly used in problems.

Each member of a set is called an element. We would be more importantly working with the number of elements in a particular set and if a set is denoted as A , the number of elements of it will be denoted as $n(A)$.

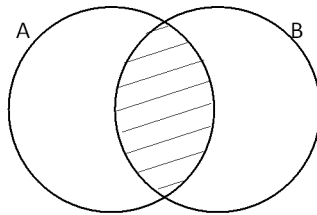
A set is usually denoted pictorially by a circle. All the elements of the set are enclosed in the circle.

Deterministic Case of Two sets

Intersection of two sets

The intersection of two sets is the set of elements that are common to both sets.

Pictorially the intersection of two sets is the shaded region as shown below:



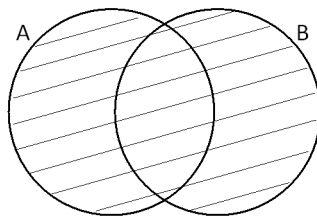
The intersection of the set of people who drink tea and the set of people who drink coffee is the set of people who drink both tea and coffee

Intersection of two sets A and B is denoted by $A \cap B$ in set terminology and its equivalence in language is set A and B .

Union of two sets

The union of two sets is the set of elements that belong to either of the two sets. Thus they include members that belong to only set A or only set B and also those that belong to both sets, the intersection.

Pictorially this is the shaded region:



The union of two sets A and B is denoted by $A \cup B$ in set terminology and its equivalence in language is set A or B .

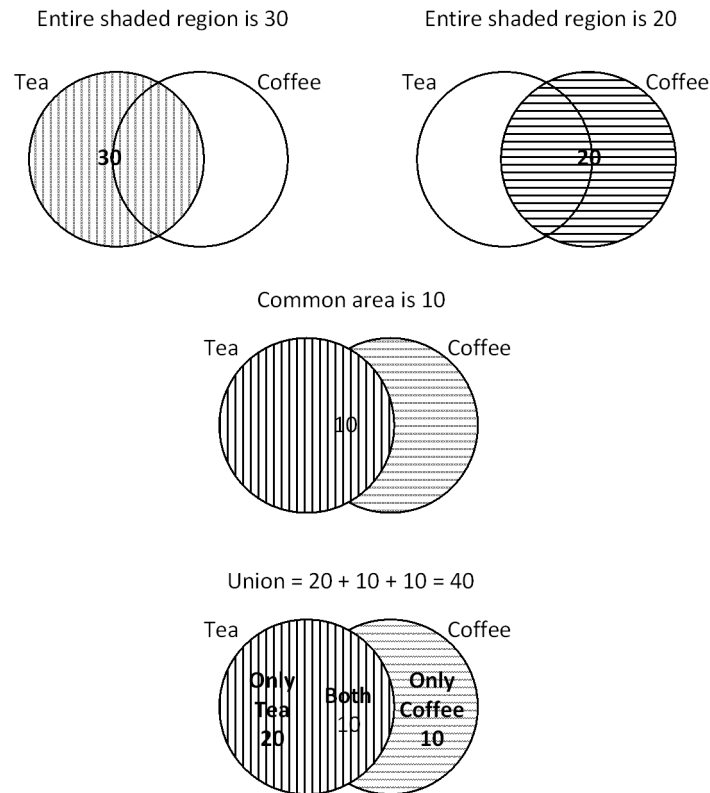
In finding the number of elements in $A \cup B$, we count all those present in the intersection of the two sets only once and not twice. In fact this is the funda on which all problems of set theory are based as explained below:

Let's say there are 30 people who drink tea and 20 people who drink coffee. Among them 10 are such that they drink both tea and coffee.

If the set of tea-drinkers is A and set of coffee drinkers is B, its obvious that $n(A \cap B) = 10$ as directly given in the question.

But what is $n(A \cup B)$? Is it number of tea drinkers + number of coffee drinkers i.e. $30 + 20 = 50$?

Not really, as you must have identified by now that when we add $30 + 20$, the 10 people who are the intersection is counted twice. Thus for the correct number of tea or coffee drinkers we must subtract 10 from 50 as we have counted them twice, to get the correct answer as 40. If this is not clear by the explanation provided above, refer to the following diagram:



Thus the formula you would have to use in almost any problem on set theory based on two sets is:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Difference between Union and Sample Space

One last thing remains to be learnt: while most of the time our focus is on the union and intersection, there is one more set that is of importance, the set of elements that belong to neither of the two sets. This brings us to the funda of a Sample Space.

There is a difference between the following two sentences:

Statement 1: Out of 100 people who drink atleast one of tea or coffee, 70 drink tea and 50 drink coffee.

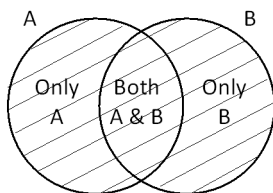
Statement 2: Out of 100 people, 70 drink tea and 50 drink coffee.

In statement 1, 100 is the union of the set of tea drinkers and coffee drinkers. Thus these 100 will be composed of those who only tea, those who drink only coffee or those who drink both.

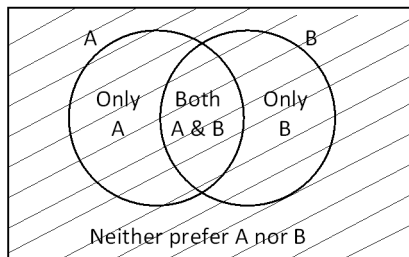
In statement 2, 100 is not the union but it is the total number of people or more appropriately the sample space. The difference is that few of these 100 people could not be drinking either of the two drinks. Say, 10 of them did not drink tea nor coffee. Then the union of the tea drinkers and coffee drinkers would be 90 and not 100.

The difference between the two statements can also be understood from the following diagram:

Statement 1: The shaded area is 100
Or $A \cup B = 100$



Statement 2: The shaded area is 100
 $A \cup B$ could be less than 100



Thus in any question make sure that you do not confuse between the sample space and the union.

The union is usually expressed in any of the following language:

100 is the number of people drinking atleast one of the drinks $\Rightarrow A \cup B = 100$

100 is the number of people who drink tea or coffee $\Rightarrow A \cup B = 100$

Of 100 people, 10 do not drink tea nor coffee... $\Rightarrow A \cup B = 90$

E.g. 1: In a school of 60 students, each student has to choose atleast one elective out of French and German. 40 students chose French and 30 chose German. How many students chose both French and German?

Since each student has to choose atleast one of the elective, 60 is the union of set of students choosing French or German.

$$\text{Hence } 60 = 40 + 30 - n(F \cap G)$$

$$n(F \cap G) = 70 - 60 = 10.$$

Thus 10 students chose both French and German

E.g. 2: In a club of 80 members, 45 play cricket, 30 play football and 10 play both cricket and football. How many do not play either cricket or football?

$$n(C \cup F) = n(C) + n(F) - n(C \cap F) = 45 + 30 - 10 = 65.$$

Thus remaining $80 - 65 = 15$ do not play either of the game.

E.g. 3: In a survey done across 350 households, 270 subscribed to Times of India, 120 subscribed to Indian Express and 30 did not subscribe to either. Find the number of households who subscribed to only Times of India.

Out of 350 households, 30 did not subscribe to any newspaper. Thus the union of those subscribing to Times of India or Indian Express is $350 - 30 = 320$.

Using $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, we have

$$320 = 270 + 120 - n(A \cap B)$$

$$n(A \cap B) = 390 - 320 = 70$$

Thus there are 70 households who subscribe to both Times of India and Indian Express. Removing these households from those subscribing Times of India will give us the number of households who subscribe to only Times of India.

Thus those subscribing to only Times of India = $270 - 70 = 200$.

Try solving all questions using the formula as explained above. Avoid using Venn diagrams as they consume more time and are used here just for explanation purpose and not for solving the questions.

Exercise

- In a gathering it was found that 60% of the people present were more than 50 kgs in weight and 70% of the people were more than 5 feet tall. If 10% of the people satisfied neither of the two conditions, find the percentage of people who were both more than 50 kgs in weight and more than 5 feet tall.

(1) 30% (2) 40% (3) 60% (4) Cannot be determined
- In a class of 60 students, 40 students sing and 45 of them dance. If all students participate in atleast one of singing or dancing, find the number of students who participate in only one of the activity.

(1) 25 (2) 30 (3) 35 (4) Cannot be determined
- In a group, 40 people drink only tea, 30 drink only coffee and 90 drink atleast one of tea or coffee. If there were 10 who drink neither tea nor coffee, what percentage of the people present drink both tea and coffee?

(1) 20% (2) 30% (3) 40% (4) Cannot be determined
- In a group of 50 people, 30 do not drink tea and 20 do not drink coffee. If 15 of them drink both tea and coffee, how many drink neither tea nor coffee?

(1) 10 (2) 15 (3) 20 (4) 25
- In The World School, a student has the option to leave out Math or Science as a subject, but he cannot leave both. After 50 students dropped Math and 30 dropped Science, there were 20 students who opted studying both the subjects. Find the strength of the class.

(1) 75 (2) 80 (3) 90 (4) 100
- Shyam visited Ram on vacation. In the mornings, they both would go for yoga. In the evenings they would play tennis. To have more fun, they indulge only in one activity per day, i.e., either they went for yoga or played tennis each day. There were days when they were lazy and stayed home all day long. There were 24 mornings when they did nothing, 14 evenings when they stayed at home, and a total of 22 days when they did yoga or played tennis. For how many days did Shyam stay with Ram?

(1) 30 (2) 28 (3) 26 (4) Cannot be determined



Deterministic Case of Three sets

One of the first things to learn in the scenario of three sets is the language used. See the following figure very carefully to understand all the nuances of the language used...Let, A be the set of people playing cricket, B be the set of people playing football and C be the set of people playing hockey.

See diagram on the following page to understand the language used.

In terms of set theory, the various notations are as explained below...

$A \cap B$: is the set of people who play cricket and football. Please note that this set includes people who play all three games. To exclude those who also play hockey, the language used is 'play *only* cricket and football'. Look at the first two pictures in the above diagram to understand the pictorial difference.

$B \cap C$: is the set of people who play football and hockey. Rest of the details is exactly as the above.

$A \cap C$: is the set of people who play cricket and hockey.

$A \cap B \cap C$: is the set of people who play all three games.

$A \cup B \cup C$: is the set of people who play atleast one game OR the set of people who play cricket or football or Hockey OR (the total number of people – the number of people who play none of cricket, football or hockey). Any of the above methods may be used to give the data regarding the union of the three sets.

The formula for the union in the case of three sets is:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Logic

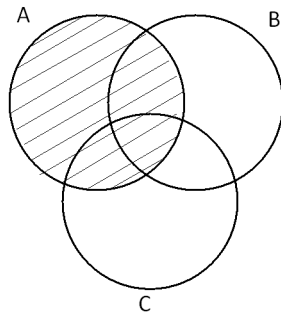
While one can learn the above formula, its better to understand the logic and then one would not need to remember...

To account for all the people who play atleast one game, we add the number of people who play cricket, the number of people who play football and the number of people who play hockey. This is $n(A) + n(B) + n(C)$.

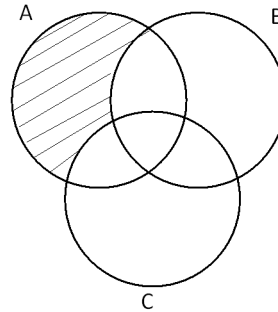
However in doing so, all those who play cricket *and* football were included in the set of those who play cricket and also in the set of those who play football and hence have been counted twice. Thus to correct the mistake, we should be subtracting the number of people who play cricket and football from the just found sum. This holds true for even those who play football and hockey or those who play cricket and hockey. Thus, after eliminating the double counting, we arrive at $n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C)$.

One should not stop here. Going one step deeper, let's look at how many times have we counted those who play all the three games. These people are a part of set A, B and C and hence, in $n(A) + n(B) + n(C)$ they were counted a total of three times. But then, these people were also a part of each of $A \cap B$, $B \cap C$, $A \cap C$. And thus when they are subtracted from the former, in effect all those who play all three games are excluded. Thus to account for this under-counting, we need to add $n(A \cap B \cap C)$ to correctly arrive at the union.

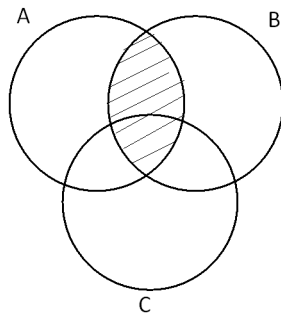
Set of people preferring A is the entire circle A.



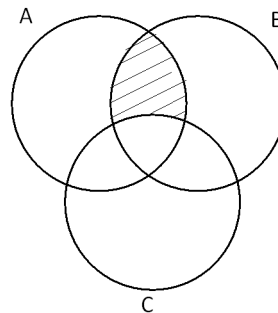
If we want to refer to the shaded area, we use 'set of people preferring *only A*'



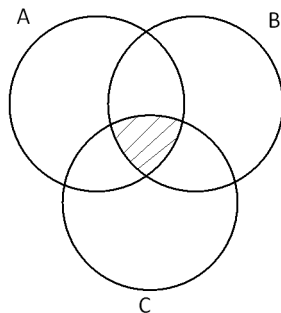
Set of people preferring A and B is the entire shaded area shown below, $A \cap B$



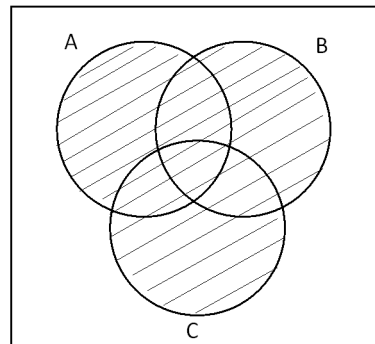
If we want to refer to the shaded area, we use "set of people preferring *only A and B*"



Set of people preferring all three, A, B and C is the entire shaded area shown below, $A \cap B \cap C$



The 'union $A \cup B \cup C$ ' or 'people preferring A or B or C' or 'those preferring atleast one' is the entire shaded area



The rectangle represents all the people i.e. the sample space. The area outside the shaded portion i.e. outside the union represents people who play none of the three sports.



It's important to understand the above logic (given in the box), because then you can form such equations for most of the common scenarios yourselves and need not mug up the following relations...

$$\begin{aligned} &\text{Number of people who play exactly two sports} \\ &= n(A \cap B) + n(B \cap C) + n(A \cap C) - 3 \times n(A \cap B \cap C) \end{aligned}$$

Since those who play all three sports are part of each of $A \cap B, B \cap C, A \cap C$, they have been counted thrice, whereas they should not be counted even once because we want only those who play two sports. Thus they are subtracted thrice.

$$\begin{aligned} &\text{Number of people who play atleast two sports} \\ &= n(A \cap B) + n(B \cap C) + n(A \cap C) - 2 \times n(A \cap B \cap C) \end{aligned}$$

This time we are looking for those who play *atleast* two games and hence we should be counted all those who play all three games, but only once and not thrice. Hence we have subtracted twice their number.

The best way to find the number of people playing exactly one sport is to subtract the number of people playing atleast two sports from the union. The formula for union and those who play atleast two sports has already been mentioned above. Alternately, since people belonging to each of $A \cap B, B \cap C, A \cap C$ have been counted twice in $n(A) + n(B) + n(C)$ and since we do not need to count them while considering those who play exactly one sport, hence we have to subtract them twice. Next, working one more level deeper, all those belonging to $A \cap B \cap C$ are counted thrice in $n(A) + n(B) + n(C)$ and when we subtract each of $A \cap B, B \cap C, A \cap C$ twice, they are subtracted a total of 6 times. Thus, to avoid this under-counting, we have to add them back thrice. Thus,

$$\begin{aligned} &\text{Those who play exactly one game} = \\ &n(A) + n(B) + n(C) - 2(n(A \cap B) + n(B \cap C) + n(A \cap C)) + 3 \times n(A \cap B \cap C) \end{aligned}$$

Lastly, if we have to find the number of people playing only cricket, it should not be too difficult to arrive at the formula $n(A) - (n(A \cap B) + n(A \cap C)) + n(A \cap B \cap C)$. This is because we do not want those who play cricket and hockey, nor do we want those who play cricket and football. But when we subtract, these two sets, we have subtracted those who play all three games twice and counted them just once in $n(A)$. Thus, to correct our under-counting, we need add $n(A \cap B \cap C)$ back.

The above should be enough to algebraically solve any question on sets theory, even without resorting to Venn diagrams. Just in case you are unable to use the above algebraic relations, you could always draw the Venn diagram and try to reason out, Venn diagrams are always easier, though may consume a little more time.

E.g. 4: In a school having 150 students, 80 play cricket, 60 play football and 50 play hockey. 30 play cricket and football, 20 play football and hockey and 10 play cricket and hockey. If 5 play all three games, find the number of students who do not play any of the three games.

The number of students not playing any of the three games is given by $150 - \text{union}$. And the union can be found out as
 $(80 + 60 + 50) - (30 + 20 + 10) + 5 = 135$.

Thus, the number of students not playing any game is $150 - 135 = 15$.

E.g. 5: Of 140 people who drink atleast one of tea, coffee or pepsi, 70 drink exactly two drinks but not the third. If the number of people who drink tea is 100 and the number of people who drink coffee is 80 and the number of people who drink pepsi is 50, find the number of people who drink all three.

Using the formula for union, we have,

$$140 = (100 + 80 + 50) - (n(T \cap C) + n(C \cap P) + n(T \cap P)) + n(T \cap C \cap P) \text{ i.e.}$$

$$(n(T \cap C) + n(C \cap P) + n(T \cap P)) - n(T \cap C \cap P) = 90 \quad \dots\dots(i)$$

Also since, there are 70 who drink exactly two of the beverages, we have

$$(n(T \cap C) + n(C \cap P) + n(T \cap P)) - 3 \times n(T \cap C \cap P) = 70$$

$$\Rightarrow (n(T \cap C) + n(C \cap P) + n(T \cap P)) = 70 + 3 \times n(T \cap C \cap P) \quad \dots\dots(ii)$$

Thus, by substitution from (ii) in (i) we have,

$$70 + 3 \times n(T \cap C \cap P) - n(T \cap C \cap P) = 90 \Rightarrow 2 \times n(T \cap C \cap P) = 20$$

Thus, the number of people who drink all three beverages is 10.

E.g. 6: Telsyn Ltd. has three projects, A, B and C in hand and there are total of 34 employees, each of whom is working on atleast one of the three projects. The number of people who are working on only A and B is 12, on only B and C is 10 and on only A and C is 6. The number of people working on A is 20 and the number of people working on only one project is the same for all the three projects. Find the number of people working on all three projects.

If the number of people working on all the three projects is x , then the number of people working on only A is $20 - 12 - 6 - x$ i.e. $2 - x$. This number is the same for those working on only B and only C.

Thus, those working on B or C but not A is $(2 - x) + (2 - x) + 10 = 34 - 20$ i.e. $14 - 2x = 14$ i.e. $x = 0$. Thus, the number of people working on all three projects is 0.

Exercise

7. In a school, 120 students play cricket, 90 play football and 75 play hockey. Further, 50 play cricket and football, 40 play cricket and hockey and 25 play football and hockey. If the total number of students in the school are 200 and if 20 of them do not play any of the three games, find the number of students who play all three games.
- (1) 10 (2) 15 (3) 25 (4) Cannot be determined
8. In a group of 100 people who subscribe to atleast one of Business Today, Business Week or Business India, 40 subscribe to exactly one of the magazine. If 20 subscribe to Business Today and Business Week, 30 subscribe to Business Week and Business India and 24 subscribe to Business Today and Business India, find the number of people who subscribe to all the three magazines.
- (1) 7 (2) 14 (3) 20 (4) Cannot be determined
9. In a class where everyone can play atleast one of cricket, hockey and football, 37% play atleast two of three games, 15% play only cricket, 23% play hockey and football, 20% play cricket and hockey and 12% like all three. If the number of students who play hockey is twice those who play football, what percentage of people play hockey or cricket?
- (1) 85% (2) 91% (3) 93% (4) Cannot be determined
10. In a group of 100, 50 are having an umbrella, 60 have a hat and 80 have sunglasses. 70 are such that they don't have both – an umbrella and a hat. Similarly 50 are such that they do not have both hat and sunglasses and 60 do not have both umbrellas and sunglasses. If there are 5 who do not own any of the three items, find the number of people who own all three items.
- (1) 20 (2) 25 (3) 30 (4) Cannot be determined
11. In a survey done, the ratio of the number of people who prefer only A, those who prefer only B and those who prefer only C is 2 : 1 : 3. The ratio of the number of people who prefer only B and A, those who prefer only A and C and those who prefer only B and C is 1 : 2 : 2. Also the ratio of the number of people who prefer only A and C to those who like only B is 2 : 3. If the total number of people who prefer atleast one of A, B and C is 30, find the number of people who prefer all three.
- (1) 7 (2) 8 (3) 9 (4) Cannot be determined
12. Among the 125 participants in a cultural festival, 45 could not sing, 55 could not dance and 75 could not play any musical instrument. Also 30 could sing and dance, 40 could dance and play a musical instrument and 30 could sing and play a musical instrument. If there were 105 participants who could NOT perform atleast one of singing, dancing or playing a musical instrument, find the number of participants who could not sing nor dance nor play a instrument.
- (1) 20 (2) 10 (3) 5 (4) Cannot be determined

The non-deterministic case – minimising or maximising the intersection/union

More interesting and tougher questions are possible in set theory when the total number of people (and not the union) is given. In such questions atleast two terms of

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) \text{ or}$$

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

are unknown and thus the unknown cannot be uniquely determined. But then a range of possible values can be found for them. To understand this more thoroughly, let's take examples

E.g. 7: Out of 80 students in a class, 50 choose math as their elective and 60 choose biology as their elective. What is the maximum and minimum number of people who have chosen both math and biology?

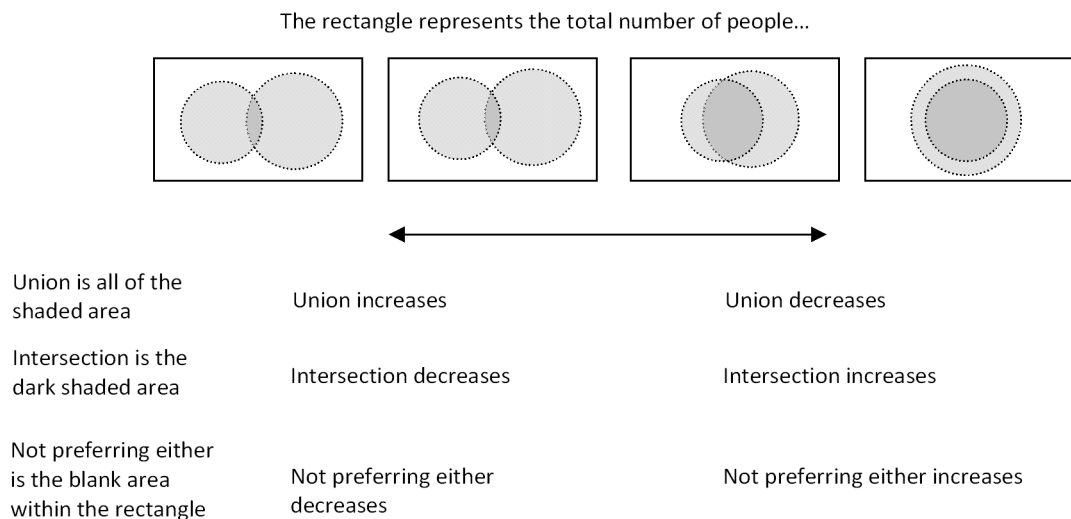
$$\text{We have the relation, } n(M \cup B) = 50 + 60 - n(M \cap B).$$

But, in this case, $n(M \cup B)$ is not necessarily 80 as we do not know if all 80 had to choose atleast one of math or biology. Nor do we know how many did choose neither math nor biology. Thus, we do not know $n(M \cup B)$ uniquely and hence cannot find $n(M \cap B)$ uniquely. However we know that the maximum value of $n(M \cup B)$ is 80. Thus, we can find the range of values that $n(M \cap B)$ can assume.

In the relation $n(M \cup B) = 110 - n(M \cap B)$, when $n(M \cup B)$ is maximum, in that case $n(M \cap B)$ will be minimum. Thus, minimum value of $n(M \cap B) = 110 - 80 = 30$.

Further $n(M \cap B)$ is a part of both the groups, those who take math, 50, and also those who take biology, 60. Thus, the maximum value that it can take is the minimum of 50 and 60 i.e. 50. This is because since it is a part of the groups, it cannot be greater than the groups themselves. And when $n(M \cap B)$ is maximum, then $n(M \cup B)$ is minimum. Thus maximum students who can take both the electives is 50 and in this case, $n(M \cup B) = 110 - 50 = 60$.

In, the case of two Venn, one need not proceed so mathematically, one can solve the questions even logically. Look at the following pictorial representation...moving left to right, we are maximizing the intersection.



From the above figure it should be obvious how any two out of the union, intersection and those preferring neither behave when one of them changes. For example when those preferring neither increases (happens when the two circles move closer to each other) then the union decreases and the intersection increases.

While with the help of the above logic, solving the above question should become very easy, there is one more important fact to learn...

Minimum number of people common to two sets

As the two circles move apart, the union increases. But there is a limit to the union increasing which is the total number of people. The union cannot be more than the number of people. Thus the two circles may not be able to move apart, further than a limit. And the situation shown in the left-most picture with the two circles being totally external to each other (intersection being 0) may not be possible. This is exactly what is happening in the above question...

Since those who take math is 50 and those who take biology is 60, if the two circles are non-overlapping, then these would account for 110 students. But there are only 80 students. Thus the two circles cannot be completely external to each other. There have to be ATLEAST $110 - 80 = 30$ common to the two circles. Since it is ATLEAST, this marks the minimum number of people common to the two sets. This is an important point and irrespective of the context one should be able to immediately identify the minimum number of people common to two sets as soon as one reads any data, e.g. ...

If in a room 70% of the people have brown eyes and 80% of people weigh over 50 kgs, then it should immediately be obvious that 50% people *should* be common to the two sets i.e. ATLEAST 50% people should have brown eyes *and* weigh over 50 kgs. The reasoning in the mind should be as such: Had both the sets of people been totally

separate, they would be 150%. But since the percent of people is 100%, 50% *have to be* common to the group. The italicized “*have to be*” and the earlier used “*should*”, indicate the MINIMUM number of people common to the two sets. Also note that it is quite possible that the number of people common to the two groups could be more than this minimum limit. This is just the lower limit.

However if the percent of people having brown eyes were 55% and the number of people weighing more than 50 kgs were 40%, since $55\% + 45\% = 95\%$ which is less than 100%, it is quite possible for the two sets to be completely exclusive. Thus the minimum number of people common to the two groups is 0% in this case. In-fact in this case, the minimum number of people who do NOT have either characteristic has to be 5%. Reason: With the two circles being totally external to each other, the union is just 95% and thus those not present in either set have to be 5%. With the circles moving closer together, this number of people not belonging to either set will only increase. Thus 5% is the minimum number of people neither having brown eyes and nor weighing more than 50 kgs.

E.g. 8: In a locality having 150 households, 100 subscribe to Times of India, 30 subscribe to Indian Express and 75 subscribe to Maharashtra Times. Find the following:

- a. The minimum number of households subscribing to Times of India and Maharashtra Times
- b. The minimum number of households subscribing to Times of India and Indian Express
- c. The minimum number of households who subscribe to NEITHER Indian Express NOR Maharashtra Times

Solution:

- a. Keeping the Times of India and the Maharashtra Times subscribers totally exclusive of each other, they would account for $100 + 75 = 175$ households. But then there are only 150 households. Thus, the two groups cannot be completely exclusive and there has to be 25 households subscribing to both Times of India and Maharashtra Times. Since they could also be more than 25, 25 is the minimum number of households subscribing to the two newspapers.
- b. Keeping the Times of India and the Indian Express subscribers totally exclusive of each other, they would account for $100 + 30 = 130$ households. And this is quite possible because there are 150 households. Thus the two groups could be totally exclusive and the minimum number of people subscribing to both Times of India and Indian Express would be 0.



c. As the two sets of those who subscribe to Indian Express and Maharashtra Times become more and more exclusive to each other i.e. move apart, the number of people who lie outside both these sets becomes lesser and lesser. Thus to minimize those who subscribe to neither of the two papers, we have to make the two as exclusive as possible. If there are no common people to the two sets, they would account for $30 + 75 = 105$. This is limiting case and in this case there would be 45 who do not subscribe to either of the papers. Thus the minimum number of people who subscribe to neither Indian Express nor Maharashtra Times is 45.

E.g. 9: In a Grand Prix, 90% of the drivers faced tyre problems, 80% of the drivers faced engine problems, 75% of the drivers faced steering problems, and 70% of the drivers faced transmission problems. Find the minimum percent of drivers who faced all four problems.

The 90% drivers who faced tyre problems and the 80% who faced engine problems cannot be different sets of drivers, as then they would number 170% of drivers. Thus, 70% of drivers *have to* face both tyre and engine problems.

Considering this as a set, this set of 70% drivers and the 75% drivers who faced steering problems, again cannot be different and 45% of them *have to be* common i.e. minimum 45% of drivers faced all of tyre, engine and steering problems.

Considering these 45% drivers as a new set, can these 45% and the 70% who faced transmission problems be exclusive sets i.e. not have any common elements? Since they would then account for $45 + 70 = 115\%$, it is not possible and 15% *have to be* common to the two sets. Thus, minimum 15% of drivers would have faced all four problems.

E.g. 10: In the above, question if it is known that 10% of the drivers faced none of the above mentioned four problems, what would have been the answer to the above question?

In this case, the union of any two sets cannot be more than 90%. Thus this time all comparisons have to be made not with 100% but with 90%.

Thus, minimum of $(90 + 80) - 90 = 80\%$ drivers had tyre and engine problems.

Minimum of $(80 + 75) - 90 = 75\%$ drivers had tyre, engine and steering problems.

And minimum of $(75 + 70) - 90 = 55\%$ drivers had all the four problems.

Case of Three Sets

In the case of three sets, the above is not as simple and nor can we visualize the Venn as conveniently as above. Thus the best way to tackle these questions is to use the algebraic relations.

Case 1: When along with the total number of people and $n(A)$, $n(B)$ and $n(C)$, the number of elements $n(A \cap B)$, $n(B \cap C)$, $n(A \cap C)$ are also given.

Using the algebraic equations, say $n(A) + n(B) + n(C) - (n(A \cap B) + n(B \cap C) + n(A \cap C)) = K$.

Then we would have $n(A \cup B \cup C) = K + n(A \cap B \cap C)$

Thus, the case of maximum intersection of all three sets will occur when the union is maximum. Two conditions provide the upper limit...

1. The union cannot exceed the total number of people; and
2. $n(A \cap B \cap C)$ cannot exceed the least of $n(A \cap B)$, $n(B \cap C)$, $n(A \cap C)$, because it is a part of each of these three groups.

One would have to check that neither of the condition is violated.

E.g. 11: In a school having 400 students, 200 play cricket, 250 play football and 150 play hockey. If 100 play cricket and football, 80 play football and hockey and 120 play cricket and hockey, find the following:

- a. The minimum number of people who do not play any of the three sports.
- b. the minimum number of people who play all three sports.

Solution:

a. The number of people not playing any of the three games will be least when the union is maximum. It would be too hasty to conclude that the maximum union would be 400 and thus the minimum number of students not playing any of the games is 0. While the union cannot exceed 400, it is not necessary that it can assume value equal to 400. The maximum could also be lower than that...

For the union, we have the following relation:

$$n(C \cup F \cup H) = (200 + 250 + 150) - (100 + 80 + 120) + n(C \cap F \cap H)$$

$$\text{i.e. } n(C \cup F \cup H) = 300 + n(C \cap F \cap H).$$

Now the maximum value that $n(C \cap F \cap H)$ can assume is the least of (100, 80, 120) as it is a part of these three groups and can atmost be equal to the least of these groups.

Thus, maximum union = $300 + 80 = 380$.

Thus the minimum number of students not playing any of the three games is 20.

b. To find the minimum number of students playing all three games, we have find the minimum number of people that *have to be* common to all possible groups of two out of $C \cap F, F \cap H, C \cap H$. A person common to $C \cap F$ and $F \cap H$ has to be playing all three games. Like-wise for any pair.

If the group of $C \cap F$ and $C \cap H$ are totally exclusive, then they would account for $100 + 120 = 220$ students. This means that there will be 220 students playing cricket, the game common to these two groups. But there are only 200 students who play cricket. Thus, 20 students *have to be* common to the two groups $C \cap F$ and $C \cap H$ i.e. a minimum of 20 has to play all three games.

This working has to be done for all pairs out of $C \cap F, F \cap H, C \cap H$ and then the maximum number of people that *have to* play all the three sports will be the minimum number of people that play all three sports.

Considering the pair, $C \cap F$ and $F \cap H$, if the two groups are exclusive, they would account for $100 + 80 = 180$ students. Since there are more than these many students who play the common sport to the groups i.e. football, the two groups can be totally exclusive and thus this pair results in minimum 0 students playing all three games.

Considering the pair, $C \cap H$ and $F \cap H$, the two groups being exclusive implies that there are $120 + 80 = 200$ students. Comparing this number with the number of students playing hockey, 150, we see that these two groups *have to have* atleast 50 common students. Thus, this pair tells us that there are a minimum of 50 students playing all three games.

Considering the highest of the three results (minimum playing all three is found to be 20, 0 and 50 among the three pairings), the minimum number of students playing all three sports has to be 50.

This, result also tells us that the minimum union has to be $300 + 50 = 350$ and that the maximum number of students who do not play any of the three sports is $400 - 350 = 50$.

If the above two questions in the example were clear, you should get the knack of working on similar lines even if the questions seems not doable by the algebraic relations, as shown in the following example...

E.g. 12: In a survey done across 115 people about their preferences about three brands, viz. A, B and C, the following results were found. 80 prefer A, 60 prefer B and 50 prefer C. Further 30 prefer A and B, 40 prefer B and C and 25 prefer A and C. Find the minimum and the maximum number of people who prefer

- a. only A
- b. only B and C

It would be a good idea to find the minimum and maximum number of people who prefer all three brands even before beginning to solve the questions.

To find maximum number of people preferring all three, we need to look at two conditions..

$$n(A \cup B \cup C) = (80 + 60 + 50) - (30 + 40 + 25) + n(A \cap B \cap C) \text{ i.e.}$$

$n(A \cup B \cup C) = 95 + n(A \cap B \cap C)$. And since the union cannot exceed 115, the intersection cannot be greater than 20.

The second condition is that the intersection cannot exceed the least of (30, 40, 25).

Thus the maximum number of people who prefer all three brands is 20

To find the minimum number of people who prefer all three we need to compare every possible pair out of $A \cap B, B \cap C, A \cap C$ with A, B and C.

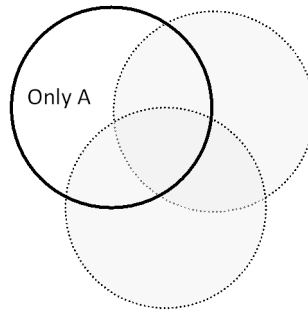
Since $A \cap B = 30$ and $B \cap C = 40$, but there are only 60 people preferring B, hence 10 people have to be common to $A \cap B$ and $B \cap C$ i.e. have to prefer all three.

Similarly the pair of $A \cap B$ and $A \cap C$ compared with those who prefer A, gives a result of minimum of 0 preferring all three brands. And the pair of $B \cap C$ and $A \cap C$ compared with those who prefer C, gives a result of minimum of 15 preferring all three brands.

Thus, the minimum number of people who prefer all three has to be 15.

Thus, the number of people who prefer all three brands would range from 15 to 20, both inclusive.

a. To solve this question our universe should be limited to only the sample space of those who prefer A. Within these people are three groups, viz. those who prefer only A, and two groups with common elements i.e. those who prefer A and B and those who prefer A and C.



To minimize the number of people who prefer only A, we would need to maximize the union of $A \cap B$ and $A \cap C$. This union will be maximized when the number of people preferring all three is minimum. Thus the maximum union of $A \cap B$ and $A \cap C$ is $30 + 25 - 15 = 40$. And the minimum number of people who prefer only A will be $80 - 40 = 40$.

To maximize the number of people who prefer only A, the union of $A \cap B$ and $A \cap C$ has to be minimized which in turn means the number of people preferring all three to be maximized. Thus the minimum union of $A \cap B$ and $A \cap C$ is $30 + 25 - 20 = 35$ and the maximum number of people who prefer only A is $80 - 35 = 45$.

b. This is a very straight-forward one, once we have found the maximum and minimum limits to the number of people who prefer all three brands.

In this case we have to just look at the 40 people who prefer B and C. Out of these 40 people, any number from 15 to 20 would also prefer C. Thus the minimum number of people who prefer only B and C is $40 - 20 = 20$ and the maximum number of people who prefer only B and C is $40 - 15 = 25$.

Case 2: Only the union and $n(A)$, $n(B)$ and $n(C)$ are given. We are free to choose $n(A \cap B)$, $n(B \cap C)$, $n(A \cap C)$ to help our case of maximizing or minimizing.

Consider this with an example...

E.g. 13: Out of 80 people who speak at least one of English, French and Spanish, 65 speak Spanish, 60 speak French and 55 speak English.

- Find the maximum number of people who speak all three languages.
- Find the minimum number of people who speak all three languages.

a. It is wrong to say that the maximum number of people who can speak all three languages will be the least of (65, 60, 55) i.e. 55. Because with this being the number of people who speak all three language, we would have

$$80 = (65 + 60 + 55) - (A \cap B + B \cap C + A \cap C) + 55$$

$$80 = 235 - (A \cap B + B \cap C + A \cap C) \Rightarrow (A \cap B + B \cap C + A \cap C) = 155$$

But then, $(A \cap B + B \cap C + A \cap C)$ has to be more than $3 \times 55 = 165$ since those who can speak all three, 55 are present in each of $A \cap B, A \cap C, B \cap C$.

While the above is too mathematical, look at it logically...

There are people who speak exactly one language, people who speak exactly two languages and people who speak all three languages. Let's say they number a, b and c respectively.

Now, since there are 80 individuals, we have $a + b + c = 80$.

But the total of the English speakers, French speakers and Spanish speakers is $65 + 60 + 55 = 180$ (these are more than the number of individuals because there is a lot of double counting in this). Thus, $a + 2b + 3c = 180$.

Subtracting the two equations, we have $b + 2c = 100$. Since we want to maximize, c , we can take $b = 0$ which gives us the maximum number of people who speak all three languages to be 50.

Alternate Logical Approach

An even more logically lucid and not involving maths even to the above extent is...

Let's call each combination of an individual and each language spoken by him to be an 'instance'. Thus if I spoke English and also French, these would be 2 'instances'

Now we have 80 individuals and a total of 180 instances. Obviously since the 80 individuals speak atleast 1 language, 80 instances are accounted by these. Now we have to account for the rest of the 100 instances. Since we want to maximize the number of persons who speak all three languages, these 100 instances can be possible because of 50 individuals speaking the remaining two languages as well.

b. To minimize the number of individuals who can speak all the three languages we can follow exactly the same process as of e.g. 9.

The 65 individuals speaking Spanish and the 60 individuals speaking French, if considered separate individuals, account for 125 individuals. But there are only 80 individuals. Thus 45 of them have to be common i.e. speak Spanish and French.

These 45 individuals and the 55 individuals speaking English, also must have $(45 + 55) - 80 = 20$ common individuals i.e. atleast 20 individuals must speak all three languages.

Alternately, using the logic of instances, we have 100 extra instances to be accounted for. 80 of these can be accounted for by each of the 80 speaking one more additional language. Yet there would be 20 instances left-over i.e. a minimum of 20 individuals have to speak all three languages.

The above type of question can be extended to any number of sets...

E.g. 14: In a group of 120 students, the number of people who can play the guitar, keyboard, drums and flute is 70, 50, 60 and 30 respectively. What is the maximum number of students who can play all four instruments, if 10 of the students can play none of the given instruments?

In this case, the union is 110 and the number of instances is 210. Thus there are 100 extra instances and to maximize the number of students who can play all four instruments, we have to assign 3 of these extra instances to each of as many individuals as possible. Thus, 33 individuals can be given 3 instances each. But our answer in this case is not 33. Because the maximum number of individuals who can play all four instruments cannot exceed the minimum of (70, 50, 60, 30). Thus the maximum number of individuals who can play all four instruments is 30 in this case.

While, the above question is answered, these 30 individuals account for only 90 of the extra 100 instances. What about the balance 10? They can easily be distributed among others in pairs or as single instances resulting in people playing three and two instruments respectively.

Exercise

13. In a gathering of 150 people, 100 drink tea and 80 drink coffee. Find the following
- The number of people who drink both tea and coffee
 (1) 30 (2) 50 (3) 80 (4) Cannot be determined
 - What is the maximum number of people who can drink both tea and coffee?
 (1) 30 (2) 50 (3) 80 (4) Cannot be determined
 - Find the maximum number of people who drink neither tea nor coffee.
 (1) 50 (2) 30 (3) 0 (4) Cannot be determined
 - Find the maximum number of people who drink exactly one of tea or coffee.
 (1) 20 (2) 120 (3) 150 (4) Cannot be determined
14. In a school, 120 students play cricket, 90 play football and 75 play hockey. Further, 50 play cricket and football, 40 play cricket and hockey and 25 play football and hockey. If the total number of students in the school is 200, what is the maximum number of students who can play all three games?
 (1) 25 (2) 30 (3) 20 (4) Cannot be determined
15. Among the people present in a room, 90% have black hair, 80% are more than 5 feet 3 inches in height and 70% are more than 55 kg in weight. Find the minimum and the maximum percentage of the people who have all the three characteristics.
 (1) 70%, 70% (2) 0%, 70% (3) 40%, 70% (4) 40%, 90%
16. 40 people prefer A, 50 people prefer B and 80 people prefer C. A and B are preferred by 30, B and C are preferred by 40 and A and C are preferred by 30. Find the minimum number of people surveyed.
 (1) 80 (2) 20 (3) 30 (4) 90
17. Out of 120 people who read atleast one of ToI, Hindu and IE, 80 read ToI, 50 read IE and 30 read Hindu. Find the maximum number of people who can read all three.
 (1) 20 (2) 30 (3) 40 (4) Cannot be determined
18. If in a group of people, 80%, 90%, 40% and 25% have watched ESPN, NATGeo, Discovery and Aastha respectively. If everyone in the group has watched atleast one of the four channels, what is the maximum percentage of people who have watched all four channels?
 (1) 25% (2) 35% (3) 45% (4) Cannot be determined



Co-ordinate Geometry

In Algebra we have seen numerous graphs of polynomial functions, e.g. graph of $ax + b$ is a line, that of $ax^2 + bx + c$ is a parabola. All these graphs were drawn with the values of x measured along the X-axis and the corresponding values of the function along the Y-axis. This method of dividing a plane into quadrants using two mutually perpendicular lines as axes (the X and Y axis) and each point on the plane being represented by two co-ordinates (x and y co-ordinates) is Co-ordinate Geometry.

For any point in the plane, the distance from the Y-axis (i.e. along the horizontal scale) is its x co-ordinate and is also called the abscissa of the point. Similarly the distance from the X-axis (i.e. along the vertical scale) is its y co-ordinate and is also referred to as ordinate of the point. And the point is represented by the ordered pair (x, y) .

Distance Formula

The distance between two points (x_1, y_1) and (x_2, y_2) is given by $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

E.g. 1: Prove that the points A (1, 1), B (-2, 7) and C (3, -3) are collinear.

$$\text{Distance AB} = \sqrt{(1 - (-2))^2 + (1 - 7)^2} = \sqrt{9 + 36} = 3\sqrt{5}$$

$$\text{Distance BC} = \sqrt{(-2 - 3)^2 + (7 - (-3))^2} = \sqrt{25 + 100} = 5\sqrt{5}$$

$$\text{Distance AC} = \sqrt{(1 - 3)^2 + (1 - (-3))^2} = \sqrt{4 + 16} = 2\sqrt{5}$$

Since $AC + AB = BC$, hence B, A and C must be collinear with A between B and C. (later we shall see how to prove the same using slopes)

E.g. 2: If two vertices of an equilateral triangle are $(0, 0)$ and $(2\sqrt{3}, 2)$, find the third vertex.

If the vertices are A $(0, 0)$, B $(2\sqrt{3}, 2)$ and C (x, y) , then since $AC = BC = AB$

$$\text{i.e. } AC^2 = BC^2 = AB^2, \text{ we have } x^2 + y^2 = (2\sqrt{3} - x)^2 + (2 - y)^2 = (2\sqrt{3})^2 + 2^2.$$

From first and last terms, we have $x^2 + y^2 = 16$ and from first and second term we have,

$$x^2 + y^2 = 12 - 4\sqrt{3}x + x^2 + 4 - 4y + y^2 \Rightarrow \sqrt{3}x + y = 4 \Rightarrow y = 4 - \sqrt{3}x$$

Substituting in $x^2 + y^2 = 16$, we have $x^2 + 16 - 8\sqrt{3}x + x^2 = 16 \Rightarrow x^2 - 4\sqrt{3}x = 0$



Thus, $x = 0$ or $4\sqrt{3}$ and correspondingly $y = 4$ or -8 .

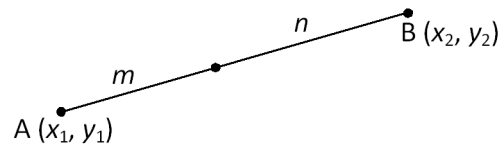
Thus the third vertex could be $(0, 4)$ or $(4\sqrt{3}, -8)$.

Section formula & Mid-point formula

Consider the line segment joining A (x_1, y_1) and B (x_2, y_2) .

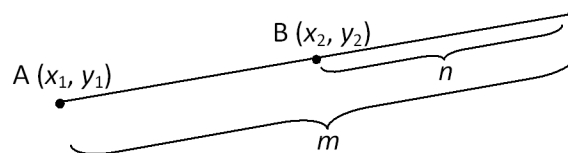
The co-ordinates of the point that divides AB internally in the ratio $m : n$ is

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right)$$



The co-ordinates of the point that divides AB externally in the ratio $m : n$ is

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n} \right)$$



A specific case of the above is to find the midpoint of AB. In this case $m : n$ is $1 : 1$ and thus the co-ordinates of the mid-point of line segment joining (x_1, y_1) and (x_2, y_2)

is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$.

Using the formula for mid-point and the section formula, one can also find the co-ordinates of the centroid of a triangle with vertices (x_1, y_1) , (x_2, y_2) and (x_3, y_3) as

$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$. This is derived by using the fact that the centroid divides

the line joining the vertex and the mid-point of the opposite side in the ratio $2 : 1$.

E.g. 3: The three vertices of a parallelogram in order are $(-2, -1)$, $(1, 0)$ and $(4, 3)$. Find the fourth vertex.

The intersection of the diagonals is the mid-point of the two diagonals. Thus the intersection of the diagonals is the mid-point of $(-2, -1)$ and $(4, 3)$ i.e.

$$\left(\frac{-2+4}{2}, \frac{-1+3}{2} \right) \text{ i.e. } (1, 1).$$

This is also the midpoint of (1, 0) and the fourth vertex, say (x, y). Thus,

$$\frac{x+1}{2} = 1 \Rightarrow x = 1 \text{ \& } \frac{y}{2} = 1 \Rightarrow y = 2$$

E.g. 4: Find the ratio in which the line segment joining (2, -3) and (5, 6) is divided by the X-axis? Also find the co-ordinate of the point at which the line segment intersects the X-axis.

Any point on the X-axis will have its *y* co-ordinate equal to 0. Thus, say the point we have to find is (x, 0) and that it divides AB in the ratio *m* : *n*, where

A is (2, -3) and B is (5, 6). Then, $\frac{6m-3n}{m+n} = 0 \Rightarrow \frac{m}{n} = \frac{1}{2}$. Thus, the required

ratio is 1 : 2 and *x* can be found from the relation $x = \frac{1 \times 5 + 2 \times 2}{1 + 2} = 3$. Thus

the required co-ordinate is (3, 0).

Slope

The slope of the line joining (*x*₁, *y*₁) and (*x*₂, *y*₂) is $\frac{y_2 - y_1}{x_2 - x_1}$.

Other aspects of a slope that needs to be kept in mind is that....

Slopes of parallel lines are equal and slopes of perpendicular lines multiply to -1.

E.g. 5: Prove that the points A (1, 1), B (-2, 7) and C (3, -3) are collinear.

Slope of AB is $\frac{7-1}{-2-1} = -2$ and slope of BC is $\frac{7-(-3)}{-2-3} = -2$. Thus AB and BC

are parallel and since B is common to the two line segments, hence A, B and C lie on a straight line.

Equation of a line

The general form of the equation of a line is $ax + by + c = 0$.

However the same equation can be expressed in many different forms, each of which can be handy in different situations...

Two Point Form:

Equation of line joining (*x*₁, *y*₁) and (*x*₂, *y*₂) is $\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$.

This form is useful when we want to find the equation of line passing through two known points...

Find the equation of line joining $(-2, 3)$ and $(5, 6)$.

The required equation is $\frac{x - (-2)}{-2 - 5} = \frac{y - 3}{3 - 6} \Rightarrow 3x + 6 = 7y - 21 \Rightarrow 3x - 7y + 27 = 0$

Slope –Point Form:

Equation of line having slope m and passing through (x_1, y_1) is $y - y_1 = m(x - x_1)$.

Consider the earlier example of finding the equation of line passing through $(-2, 3)$ and $(5, 6)$. The slope of the line could be found as $\frac{3 - 6}{-2 - 5} = \frac{3}{7}$. Then the equation can

be found as $y - 3 = \frac{3}{7}(x - (-2)) \Rightarrow 7y - 21 = 3x + 6$ which yields the same equation as above. Even if we had used the point $(5, 6)$ we would arrive at the same equation, $y - 6 = \frac{3}{7}(x - 5) \Rightarrow 7y - 42 = 3x - 15 \Rightarrow 3x - 7y + 27 = 0$

Slope –Intercept Form:

The equation of a line having slope, m and y-intercept, c is $y = mx + c$. We have extensively seen this form of the equation as the linear polynomial $ax + b$ in algebra.

Two Intercept Form:

The equation of the line having x-intercept as a and y-intercept as b is $\frac{x}{a} + \frac{y}{b} = 1$.

Slope of a line:

Consider the general form of a line, $ax + by + c = 0$. Re-writing the equation in the slope-intercept form, we get, $y = -\frac{a}{b}x - \frac{c}{b}$. Thus the slope of the line $ax + by + c = 0$

is $-\frac{a}{b}$.

The slope of the line is also defined as the tan of the angle made by the line with the positive X-axis i.e. if θ is the angle made by the line with the positive X-axis, then the slope is $\tan \theta$.

It is again worthwhile to recollect that slopes of parallel lines are equal and that the product of slopes of perpendicular lines is -1 .

E.g. 6: A line passing through $(a, 2a)$ and $(-2, 3)$ is perpendicular to the line $4x + 3y + 5 = 0$. Find the value of a .

Slope of the line $4x + 3y + 5 = 0$ is $-\frac{4}{3}$ and slope of line

passing through $(a, 2a)$ and $(-2, 3)$ is $\frac{3-2a}{-2-a}$. Since the two

lines are perpendicular, product of their slopes is -1 and thus

$$\left(-\frac{4}{3}\right) \times \left(\frac{3-2a}{-2-a}\right) = -1 \Rightarrow \frac{12-8a}{-6-3a} = 1 \Rightarrow 5a = 18 \Rightarrow a = \frac{18}{5}$$

E.g. 7: The line joining two points A $(2, 0)$ and B $(3, 1)$ is rotated about point A in the anti-clockwise direction through an angle of 15 degrees. Find the equation of the new line.

The slope of line AB is $\frac{1-0}{3-2} = 1$. The slope is also given by $\tan\theta$ where θ is

the angle made by the line with the positive X-axis. Thus, $\tan\theta = 1 \Rightarrow \theta = 45^\circ$

After rotation the new line will make an angle of $45 + 15 = 60$ degrees and its slope will be $\tan 60 = \sqrt{3}$.

Equation of line with slope $\sqrt{3}$ and passing through the point $(2, 0)$ is given by $y - 0 = \sqrt{3}(x - 2)$ i.e. $\sqrt{3}x - y - 2\sqrt{3} = 0$.

E.g. 8: The three sides of a triangle are given by $x + y - 6 = 0$, $x - 3y - 2 = 0$ and $5x - 3y + 2 = 0$. Find the equation of the median to the side represented by the second equation.

The vertices of the triangle can be found by solving pairs of equations simultaneously. Considering AB as $x + y - 6 = 0$, BC as $x - 3y - 2 = 0$ and AC as $5x - 3y + 2 = 0$, vertex A is the simultaneous solution of $x + y - 6 = 0$ and $5x - 3y + 2 = 0$ and similarly vertex B and C can be found out.

We find the vertices as A $(2, 4)$, B $(5, 1)$ and C $(-1, -1)$.

The required median is AD where D is the mid-point of BC. Thus co-

ordinates of D are $\left(\frac{5-1}{2}, \frac{1-1}{2}\right)$ i.e. $(2, 0)$.

Required equation is equation of line joining $(2, 4)$ and $(2, 0)$ and this can be

found using the two point form as $\frac{x-2}{2-2} = \frac{y-4}{4-0} \Rightarrow x-2=0$ i.e. $x = 2$.

E.g. 9: Find the mirror image of the point $(-8, 12)$ when it is reflected about the line $4x + 7y + 13 = 0$

If A is $(-8, 12)$ and B is its required reflection, then line AB will be perpendicular to $4x + 7y + 13 = 0$. Since slope of $4x + 7y + 13 = 0$ is $-\frac{4}{7}$,

slope of AB will be $\frac{7}{4}$.

Equation of line AB, using point slope form will be

$$y - 12 = \frac{7}{4}(x - (-8)) \text{ i.e. } 7x - 4y + 104 = 0$$

The point of intersection of $4x + 7y + 13 = 0$ and $7x - 4y + 104 = 0$ can be found by solving the two equations simultaneously as $(-12, 5)$. This point $(-12, 5)$ will be the midpoint of A $(-8, 12)$ and B (p, q) , say. Thus,

$$\frac{-8 + p}{2} = -12 \Rightarrow p = -16 \text{ and } \frac{12 + q}{2} = 5 \Rightarrow q = -2. \text{ Thus the required}$$

reflection is $(-16, -2)$.

E.g. 10: Find the equation of a straight line parallel to $2x + 3y + 11 = 0$ and whose sum of the x and y intercepts is 15.

Since the required line is parallel to $2x + 3y + 11 = 0$, we can assume its equation as $2x + 3y + k = 0$. Putting $x = 0$, we get the y -intercept as $-\frac{k}{3}$

and putting $y = 0$, we get the x -intercept as $-\frac{k}{2}$. And since the sum of the

intercepts has to be 15, we have $-\frac{k}{3} - \frac{k}{2} = 15 \Rightarrow k = -18$. Thus required

equation is $2x + 3y - 18 = 0$.

E.g. 11: Find the equation of the line through $(2, 3)$ so that the segment of the line intercepted between the axes is bisected at this point.

The end-points of the segment of the line intercepted between the axes can be assumed as $(a, 0)$ and $(0, b)$. One of the co-ordinates will be 0 because the two end-points will lie on the X and Y axis. Further since $(2, 3)$ is the midpoint of these two points, we can easily find $a = 4$ and $b = 6$ using formula for mid-point.

Thus required equation is that of a line passing through $(4, 0)$, $(2, 3)$ and $(6, 0)$. We could use any form to find the equation. Using the Two Intercept

form, the required equation is $\frac{x}{6} + \frac{y}{4} = 1 \Rightarrow 2x + 3y - 12 = 0$.

Other formulae

Other formulae, though used only occasionally are...

Distance between point and line

The perpendicular distance from the point (x_1, y_1) to the line $ax + by + c = 0$ is

$$\left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

Distance between two lines

The perpendicular distance between two parallel lines $ax + by + c_1 = 0$ and

$$ax + by + c_2 = 0 \text{ is given by } \left| \frac{c_1 - c_2}{\sqrt{a^2 + b^2}} \right|$$

Angle between two lines

The angle, θ , between two lines with slope m_1 and m_2 can be found using the

$$\text{formula } \tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|.$$

E.g. 12: Find the points on line $x + y = 4$ that lie at a distance of 1 unit from the line $4x + 3y = 10$.

If the required point is (a, b) , then since it lies on $x + y = 4$, we have $a + b = 4$. Since distance of point (a, b) from line $4x + 3y = 10$ is 1 unit, we

$$\text{have } \left| \frac{4a + 3b - 10}{\sqrt{4^2 + 3^2}} \right| = 1 \Rightarrow 4a + 3b - 10 = \pm 5 \text{ i.e. } 4a + 3b = 15 \text{ or } 5.$$

Solving $a + b = 4$ and $4a + 3b = 15$, we get the required point as $(3, 1)$

Solving $a + b = 4$ and $4a + 3b = 5$, we get the required point as $(-7, 11)$.

Thus there are two points on $x + y = 4$ that lie at a distance of 1 unit from the line $4x + 3y = 10$ viz. $(3, 1)$ and $(-7, 11)$.

E.g. 13: If two sides of a square lie along the lines $x + y - 1 = 0$ and $2x + 2y - 3 = 0$, find the area of the square.

The given equations of lines are that of parallel lines and thus will include two opposite sides of the square. The side of the square will then be equal to the perpendicular distance between the two lines. Expressing the two lines with same coefficients of x and y , we have the lines as $2x + 2y - 2 = 0$



and $2x + 2y - 3 = 0$ and the perpendicular distance between the lines is

$$\left| \frac{-2 - (-3)}{\sqrt{2^2 + 2^2}} \right| = \frac{1}{2\sqrt{2}} \text{ and the area of the square will be } \left(\frac{1}{2\sqrt{2}} \right)^2 = \frac{1}{8}.$$

E.g. 14: Find the equation of two straight lines that are parallel to $x + 7y + 2 = 0$ and at a unit distance from the point $(1, -1)$.

Since the lines are parallel to $x + 7y + 2 = 0$, their equation can be assumed as $x + 7y + k = 0$. Since the distance of $(1, -1)$ from the line $x + 7y + k = 0$ is

1 unit, we have $\left| \frac{1 + 7(-1) + k}{\sqrt{1^2 + 7^2}} \right| = 1 \Rightarrow \frac{k - 6}{5\sqrt{2}} = \pm 1$. Thus, the values of k could

be $6 + 5\sqrt{2}$ or $6 - 5\sqrt{2}$ and the required equations are $x + 7y + 6 + 5\sqrt{2} = 0$ and $x + 7y + 6 - 5\sqrt{2} = 0$.

E.g. 15: Find equation of straight line that passes through the origin and makes an angle of 45 degrees with the straight line $\sqrt{3}x + y = 11$.

If m_1 is the slope of the line whose equation is needed and m_2 is the slope of $\sqrt{3}x + y = 11$, since the angle between them is 45 degrees

we have, $\tan 45 = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$. We also know that $m_2 = -\sqrt{3}$. Thus,

$$\left| \frac{m_1 + \sqrt{3}}{1 - \sqrt{3}m_1} \right| = 1 \Rightarrow m_1 + \sqrt{3} = 1 - \sqrt{3}m_1 \text{ or } m_1 + \sqrt{3} = \sqrt{3}m_1 - 1$$

$$\text{Thus, } m_1 = \frac{1 - \sqrt{3}}{1 + \sqrt{3}} \text{ or } \frac{\sqrt{3} + 1}{\sqrt{3} - 1}.$$

Since the line passes through origin, its y -intercept is 0 and using slope-

intercept form, its equation can be found as $y = \frac{1 - \sqrt{3}}{1 + \sqrt{3}}x$ or $y = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}x$ i.e.

$$(1 - \sqrt{3})x - (1 + \sqrt{3})y = 0 \text{ or } (\sqrt{3} + 1)x - (\sqrt{3} - 1)y = 0.$$

Exercise

- For what value of k are the points $(k, 2 - 2k)$, $(-k + 1, 2k)$ and $(-4 - k, 6 - 2k)$ collinear?
 (1) 1 (2) $-1/2$ (3) $1/2$ (4) 1 or $-1/2$
- If two adjacent vertices of a parallelogram are $(3, 2)$ and $(-1, 0)$ and the diagonals cut at $(2, -5)$, find the other two vertices of the parallelogram.
 (1) $(1, -12)$ & $(5, -10)$ (2) $(1, -10)$ & $(5, -12)$ (3) $(-1, 12)$ & $(-5, 10)$ (4) $(-1, 10)$ & $(-5, 12)$
- If the co-ordinates of the mid-points of the sides of a triangle are $(1, 2)$, $(0, -1)$ and $(2, -1)$, find the sum of the x -co-ordinates of the vertices.
 (1) 3 (2) 2 (3) -2 (4) -3
- Find the point on the Y -axis which is equidistant from $(2, 3)$ and $(-4, 1)$
 (1) $(0, 1)$ (2) $(0, 2)$ (3) $(0, -1)$ (4) $(0, -2)$
- Find the co-ordinates of the ortho-center of the triangle with vertices $(2, -2)$, $(-2, 1)$ and $(5, 2)$
 (1) $(2, -2)$ (2) $(-2, 1)$ (3) $(5, 2)$ (4) Cannot be determined
- Find the value of k if the lines $3x - 4y - 13 = 0$, $8x - 11y - 33 = 0$ and $2x - 3y + k = 0$ are concurrent.
 (1) -3 (2) -5 (3) -7 (4) -9
- Find the equation of a line which divides the join of $(1, 0)$ and $(3, 0)$ in the ratio $2 : 1$ and is perpendicular to it.
 (1) $7x = 3$ (2) $3x = 7$ (3) $3y = 7$ (4) $7y = 3$
- Find the point/s on the X -axis whose perpendicular distance from the line $4x + 3y = 12$ is 4.
 (1) $(8, 0)$ (2) $(2, 0)$ (3) $(-2, 0)$ (4) $(8, 0)$ or $(-2, 0)$
- If the mirror image of the point $(2, 1)$ with respect to a line is $(5, 2)$, find the equation of the line.
 (1) $x + 3y = 12$ (2) $3x + y = 12$ (3) $3x - y = 12$ (4) $x - 3y = 12$
- Find the acute angle between the diagonals of the parallelogram whose vertices in order are $(2, -1)$, $(0, 2)$, $(2, 3)$ and $(2, 0)$.
 (1) 30 degrees (2) 45 degrees (3) 60 degrees (4) 90 degrees



Heights and Distances

Common Right Angle Triangles and their trigonometric ratios

In questions on heights and distances, invariably, there would be a triangle with angles of 30° , 60° , 90° or 45° , 45° , 90° . One side of the triangle will be given and the other side of the triangle will be asked.

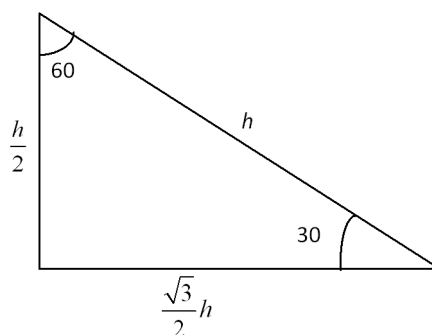
While these questions are based on the value of sine, cosine and tan of 30° , 45° and 60° , even if you are not aware of them, you can memorize the following relation between sides of the triangle and solve them. It is strongly advised that you attempt the questions on 'Heights and distance' if any of them come in the question paper as these are very easy once you memorize the following relation:

In a 30-60-90 triangle,

Side opposite to 30° is $\frac{1}{2}$ the hypotenuse

Side opposite to 60° is $\frac{\sqrt{3}}{2}$ times the hypotenuse.

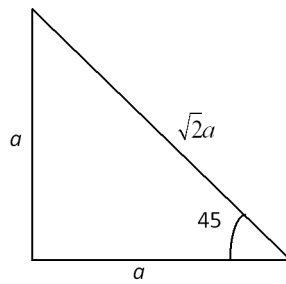
Thus if the hypotenuse is considered to be h , you should just remember the following figure:



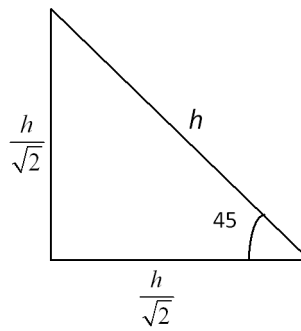
A 45-45-90 triangle is even easier...

Since it is an isosceles triangle, both the sides opposite to 45° are equal in length and are $\frac{1}{\sqrt{2}}$ times the hypotenuse. Alternately the hypotenuse is $\sqrt{2}$ times the side opposite to 45° .

If side is given as a



If hypotenuse is given as h



Alternately, one should also be conversant with the following trigonometric ratios...

In a right angle triangle,

$$\sin \theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos \theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

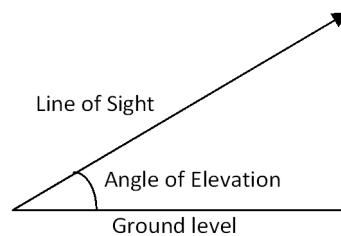
$$\tan \theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

And...

	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

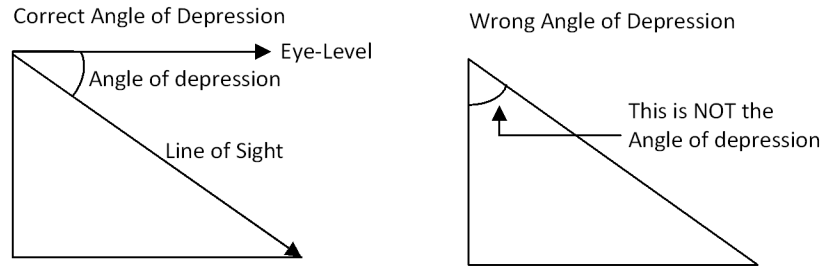
Angle of Elevation and angle of Depression

In all questions on Heights and Distances, the ground level is considered to be horizontal. When one has to look at a tower top or a pole top, one has to look upwards. The angle of elevation is the angle between the eye-level (ground-level in most of the problems) and the line of sight to the top of a tower, pole or mountain.

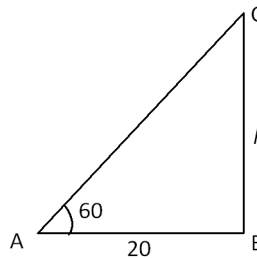


When one is standing on top of a building or mountain and viewing an object on the ground, one has to lower his line of sight from the horizontal line at eye-level. The angle between the horizontal line through the eye-level and the line of sight is the angle of depression. Many students make a mistake and consider the angle

of depression as the angle between the line of sight and the vertical tower. This is wrong (as shown in the following diagram) and you should avoid committing this kind of mistake.

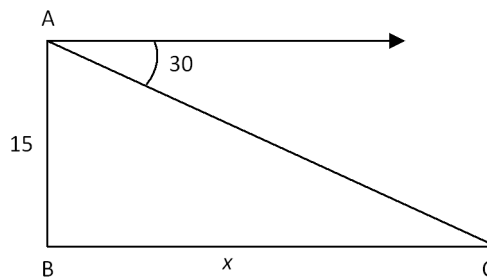


E.g. 1: The angle of elevation to the top of a tower from a point 20 ft from the base of the tower is 60° . Find the height of the tower?



Since AB is side opposite to the 30° , the hypotenuse will be twice the length of AB i.e. 40 feet. The height of the tower is the side opposite to 60° and hence will be $\frac{\sqrt{3}}{2}$ times the hypotenuse and will be $\frac{\sqrt{3}}{2} \times 40 = 20\sqrt{3}$ feet in this example.

E.g. 2: I am standing at the top of a vertical cliff that is 15 meters above ground level. The angle of depression to a point x meters from the base of the cliff is 30° . Find the value of x .

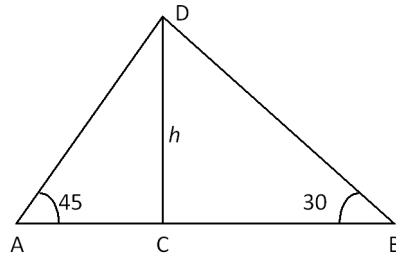


Since AB is opposite to 30° , it is half the hypotenuse.
Thus the hypotenuse, $AC = 2 \times 15 = 30$

Since x is side opposite to 60° , it is $\frac{\sqrt{3}}{2}$ times the hypotenuse i.e. it is $15\sqrt{3}$

E.g. 3: Two points A and B are on either sides of a tower such that A, base of the tower and B are in a straight line. The angle of elevation from points A and B to the top of the tower is 45° and 30° respectively. If the distance between AB is 50 meters, find the height of the tower.

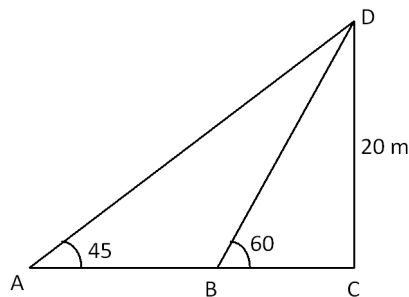
Let the height of the tower be h meters.



$$AC = h \text{ and } CB = \sqrt{3}h.$$

$$\text{Thus } h + \sqrt{3}h = 50 \Rightarrow h = \frac{50}{1 + \sqrt{3}}$$

E.g. 4: Two points A and B are on the same side of a tower such that A, B and the base of the tower are in a straight line. The angle of elevation from points A and B to the top of the tower is 45° and 60° respectively. If the height of the tower is 20 meters, find the distance AB.



In $\triangle BCD$, we know $CD = 20$ meters. Since CD is side opposite to 60° , it is $\frac{\sqrt{3}}{2}$ time the hypotenuse. Thus hypotenuse $BD = \frac{2}{\sqrt{3}} \times 20 = \frac{40}{\sqrt{3}}$.

BC is side opposite to 30° and hence is half the hypotenuse and is of length $\frac{20}{\sqrt{3}}$ meters.

In $\triangle ACD$, since CD is 20 meters, so is AC , being an isosceles triangle.

$$AB = AC - BC = 20 - \frac{20}{\sqrt{3}} = \frac{20(\sqrt{3} - 1)}{\sqrt{3}}$$

Exercise

1. The angle of depression while viewing an object from a mountain top is 60° . If the height of the mountain is 75 meters, at what distance from the base of the mountain is the object?

(1) $75\sqrt{3}$ (2) $50\sqrt{3}$ (3) $25\sqrt{3}$ (4) 150

2. A ladder of length 10 meters is leaning against a vertical wall such that the ladder makes an angle of 45° with the level ground. The base of the ladder slips away from the wall and now it is making an angle of 30° with the ground. By how many meters did the top of the ladder slide down along the vertical wall?

(1) $5(2\sqrt{2}-1)$ (2) $5(\sqrt{2}-1)$ (3) $5(\sqrt{3}-1)$ (4) $5(2\sqrt{3}-1)$

3. The angle of elevation to the top of a tower is 45° . On walking 10 meters towards the tower, the angle of elevation becomes 60° . What is the height of the tower?

(1) $5(\sqrt{3}+1)$ (2) $10(\sqrt{3}+1)$ (3) $\frac{10}{(\sqrt{3}+1)}$ (4) $15+5\sqrt{3}$

4. The angle of elevation to the top of a tower of height 50 meters is 45° . How many meters should one move backwards, so that the angle of elevation of the tower becomes 30° ?

(1) $50(\sqrt{3}-1)$ (2) $\frac{50}{(\sqrt{3}-1)}$ (3) $50-\frac{50}{\sqrt{3}}$ (4) $\frac{50}{\sqrt{3}}-10$

5. From a height of 20 meters above ground level, the angle of elevation to the top of a tower is 45° and from a height 30 meters above ground level, the angle of elevation to the top of the tower is 30° . Find the height of the tower from ground level.

(1) $30+5(\sqrt{3}-1)$ (2) $20+5(\sqrt{3}-1)$ (3) $20+5(\sqrt{3}+1)$ (4) $30+5(\sqrt{3}+1)$





Calendars

Concept of Odd days

Questions on calendar ask us to identify which day of the week a given date falls on. To identify the day of the week we use the fact that the day of the week is the same after every 7 days i.e. a week. If it is a Sunday today, 7 days from now it will again be a Sunday. Also 14 days, or 21 days or 28 days or any multiple of 7 days from now, will be a Sunday.

Let's say today is Wednesday. What will be the day 45 days from today? Now we know that 42 days from today will again be a Wednesday. The extra days over complete number of week are called 'odd days'. In this example the number of odd days is $45 - 42 = 3$. Thus the week-day 45 days from now will be 3 more days further than a Wednesday i.e. it will be a Saturday.

Thus, we would have to find the complete number of weeks from a reference point and the number of days extra than this, called 'odd days'.

The Gregorian Calendar

We follow the Gregorian calendar. The following facts about the Gregorian calendar are important to solve questions on calendar:

1. Years are divided as leap and non-leap years.
2. Non-leap years have 365 days i.e. 52 weeks and 1 odd day.

Leap years have 366 days i.e. 52 weeks and 2 odd days.

3. A year is a leap year if the year is divisible by 4. Thus each of 2004, 2008, 2012, ... will be a leap year. But...

If a year is divisible by 100, for the year to be a leap year, it should be divisible by 400. Thus while 1200, 1600, 2000 are leap years, 1700, 1800, 1900, 2100 are not leap years (even though they are divisible by 4).

4. Each of Jan, Mar, May, July, Aug, Oct and Dec has 3 odd days.

Each of Apr, Jun, Sep, Nov has 2 odd days.

Feb of a leap year has 1 odd day and Feb of a non-leap year has 0 odd days.



Standard Questions on Calendars

Questions on calendar come in two types viz.

1. Finding the day of the week for a given date when a reference date and day is given.
2. Finding the day of the week for any given date without any reference.

Type 1: When a reference is given

In this case all you need to do is find the number of odd days from the date of reference to the date for which the day has to be found out. While doing so, be careful of the leap years that come in-between.

Also in all the working done below, we would INCLUDE the date for which the week-day is required to be found but we will EXCLUDE the date for which the week-day is given. Thus, if we get 1 odd day, it will be the next week-day in comparison to the given week-day. And if we have 2 odd days, the required week-day will be the given week-day + 2. And so on. If we get 0 odd days, the day of the week will be the same week-day as given.

E.g. 1: If 10th April 2007 is a Tuesday, what day of the week will 25th December 2010 fall on?

11th April 2007 to 10th April 2008 will have 2 odd days (a leap day of Feb 2008 will be in this period)

11th April 2008 to 10th April 2009 will have 1 odd day

11th April 2009 to 10th April 2010 will have 1 odd day

In the year 2010, rest of April (20 days), May, Jun, July, Aug, Sep, Oct, Nov and upto 25th Dec (inclusive) will have $20 + 3 + 2 + 3 + 3 + 2 + 3 + 2 + 25 = 63$ i.e. 0 odd days.

Thus from 11th April 2007 to 25th Dec 2010 we would have $2 + 1 + 1 + 0 = 4$ odd days.

Thus 25th December would be 4 days after a Tuesday i.e. it will be a Saturday.

In problems like this, one should ideally go directly from 11th April 2007 to 10th April 2010 and identify the odd days as $2 + 1 + 1$ in one shot.

E.g. 2: If 14th August 1947 was a Monday, what day of the week was 26th January 1950?

From 15th August 1947 to 14th August 1949 we would have $2 + 1 = 3$ odd days.

In the rest of days of Aug 1949 to 26th Jan 1950 we would have $17 + 2 + 3 + 2 + 3 + 26 = 53$ i.e. 4 odd days.

Thus total odd days from 15th Aug 1947 to 26th Jan 1950 is $3 + 4 = 7$ i.e. 0 odd day. Since 14th August is given to be a Monday, 26th Jan 1950 will also be a Monday.

E.g. 3: If 10th April 2007 is a Tuesday, what day of the week would 2nd October 2002 have been?

While the question asks us to go backwards, a safe strategy would be to go forward from 2nd Oct 2002 to 10th April 2007 and find the odd days in this period.

From 3rd Oct 2002 to 2nd October 2006 we would have $1 + 2 + 1 + 1 = 5$ odd days.

Rest of Oct 2006 (29 days) and Nov 2006 to 10th April 2007 would give us $29 + 2 + 3 + 3 + 0 + 3 + 10 = 50$ i.e. 1 odd days.

Thus from 3rd October 2002 to 10th April 2007 there are a total of $5 + 1 = 6$ odd days.

Thus, 10th April 2007 will be 6 days of the week ahead than 2nd October 2002. And since this 6 days ahead is a Tuesday, 2nd October would be 6 days behind a Tuesday i.e. it will be a Wednesday.

Type 2: When no reference is given

In such questions you have to make do with the fact that the day on the beginning of the calendar was a Monday i.e. the date 1st Jan 0001 was a Monday.

Finding the odd number of days in a block of 100, 200, 300, 400 years

Strictly starting from the first day of the calendar,

The first 100 years (1st Jan 0001 to 31st Dec 0100) will have 24 leap years (0004, 0008, 0012,, 0096) and 76 non-leap years (note that the year 0100 is not a leap year).

Thus the first 100 years would have $24 \times 2 + 76$ odd days i.e. 124 odd days. These will amount to a total of 17 weeks and 5 odd days.

Thus first 100 years will have 5 odd days.

Similarly the next hundred years (1st Jan 0101 to 31st Dec 0200) would also have 5 odd days. Thus the first 200 years would have $5 + 5 = 10$ i.e. 3 odd days.

The next (third) 100 years (1st Jan 0201 to 31st Dec 0300) would again have 5 odd days.

Thus the first 300 years will have $3 + 5 = 8$ i.e. 1 odd day

But the fourth 100 years (1st Jan 0301 to 31st Dec 0400) would be different. It would have 25 leap years and 75 non-leap years. This is because the last year (0400) is a leap year whereas in the earlier cases the last years (0100, 0200, 0300) were not leap years.

Thus the fourth 100 years would have $2 \times 25 + 75 = 125$ i.e. 17 weeks and 6 odd days.

Thus the first 400 years would have $1 + 6 = 7$ i.e. no odd days

This fact that first 400 years does not have any odd day is used as follows:

1st Jan of the years 0001, 0401, 0801, 1201, 1601, 2001, 2401 would all be Monday

Use this fact to come within 400 years of the given date.

Once you come to a date closest to the date given then we have to go forward in blocks of 100 years.

100 years would have 5 odd days

200 years would have 3 odd days

300 years would have 1 odd day.

Once we come even closer to the given date, we would have to go forward in blocks of 4 years.

Every block of 4 year has 3 non-leap years and 1 leap year. Thus it has $3 + 2 = 5$ odd days.

After this we would have to go forward taking each year at a time.

If you follow the above strictly starting from the 1st of January 0001, (and INCLUDING 1st Jan 0001 and the given date as well) then the odd days so got would result in the day of week as follows:

0 odd days: Sunday 1 odd day: Monday 2 odd day: Tuesday

and so on till

6 odd day: Saturday.

Let's learn this with an example:

E.g. 4: Find the day of the week on 15th September 1995.

From the above stated fact, the first 1600 years would have 0 odd days.

From 1st Jan of the above mentioned years, the next,

100 years would have 5 odd days

200 years would have $5 + 5 = 10$ i.e. 0 odd days

300 years would have $5 + 5 + 5 = 15$ i.e. 0 odd days.

Use the above fact to come within 100 years of the given date.

From 1st Jan 1601 to 31st Dec 1900 is a period of 300 years and thus would have 0 odd days. You need not find the day of the week for the intermediate dates. You can simply go on collecting the odd days. Thus till 31st Dec 1900 we have collected 0 odd days.

Now, once you have reached within 100 years of the given date, move forward in blocks of 4 years till the time you reach within 4 years of the given date.

Each block of 4 years would have 5 odd days.

From 1st Jan 1901 to 31st Dec 1992 is a block of $\frac{92}{4} = 23$ four-year periods.

Thus it will have $23 \times 5 = 115$ i.e. 0 odd days. Thus odd days collected so far are $0 + 0 = 0$.

Once you reach within 4 years of the date given, we have inch forward on a year on year basis:

The year 1993 would have 1 odd day (cumulative odd days so far is $0 + 1 = 1$)

The year 1994 would have 1 odd day (cumulative odd days so far is $1 + 1 = 2$)

The year 1995 upto the date given i.e. 15th September (inclusive) would have $3 + 0 + 3 + 2 + 3 + 2 + 3 + 3 + 15$ odd days for the months of Jan, Feb, Mar, April, May, Jun, July, Aug and September respectively. These total up to 34 i.e. 2 odd days.

Thus till 15th September, 1995 we collected a total of $2 + 2 = 4$ i.e. 0 odd days.

0 odd days would imply that there is one more day than a complete week that started on Monday and ended on Sunday. Thus 0 odd days would imply the day is a Monday

Similarly 1, 2, 3, 4, 5, 6, 0 odd days would imply the day is a Tuesday, Wednesday, Thursday, Friday, Saturday and Sunday respectively.

Thus 15th September, 1995 would be a Friday.

The above example was a probably the lengthiest one that you can expect. Most other dates would not be as lengthy...

E.g. 5: Find the day of the week on 5th May, 2007.

Every 400 years from the start of the calendar would yield 0 odd days.
Thus, till 31st Dec 2000 we would get 0 odd days.

Now since the year asked is 2007, we do not have to go forward by any 100 years. So let's go forward by blocks of 4 years.

From 1st Jan 2001 to 31st Dec 2004, there would be 5 odd days.

The year 2005 would have 1 odd day.

The year 2006 would have 1 odd day.

The year 2007, till 5th May (inclusive) would have $3 + 0 + 3 + 2 + 5 = 13$ i.e. 6 odd days.

Thus from start of the calendar to 5th May 2007, we have $5 + 1 + 1 + 6 = 13$ i.e. 6 odd days.

Thus 5th May 2007 will be a Saturday.

Exercise

- If 8th March, 2005 was a Wednesday, what was the day on 8th March 2004?
(1) Tuesday (2) Wednesday (3) Thursday (4) Friday
- If 10th January, 2009 will be a Saturday, what will be the day on 10th January, 2008?
(1) Tuesday (2) Wednesday (3) Thursday (4) Friday
- If CAT is always held on the third Sunday of November and in the year 2007 it was on the 18th of November, then on what date will it be held in 2009?
(1) 15th (2) 16th (3) 19th (4) 20th
- What was the day of the week on 16th July, 1776?
(1) Monday (2) Tuesday (3) Saturday (4) Friday
- What was the day of the week on 16th April, 2000?
(1) Monday (2) Friday (3) Saturday (4) Sunday

Base System

The numbers we use in everyday calculations are invariably in the decimal system. By decimal (10) system, we imply two important facets:

1. Each position, leftwards of the decimal place, has a value that is successive power of 10 i.e. 10^0 , 10^1 , 10^2 ,

Thus, we have the unit's digit, whose positional value is 10^0 i.e. 1.

We have the ten's digit whose positional value is 10^1 i.e. 10

We have the hundred's digit whose positional value is 10^2 i.e. 100. And so on.

2. There are 10 unique digits which we know are 0, 1, 2, ..., 9.

When any of these digit appear in any position in a number, the value of that position is it's positional value multiplied with the digit in that position. Thus,

$$38 = 30 + 8 = 3 \times 10^1 + 8 \times 10^0$$

$$1265 = 1000 + 200 + 60 + 5 = 1 \times 10^3 + 2 \times 10^2 + 6 \times 10^1 + 5 \times 10^0$$

The above can also be extended to the right of the decimal place e.g....

$$456.27 = 400 + 50 + 6 + \frac{2}{10} + \frac{7}{100} = 4 \times 10^2 + 5 \times 10^1 + 6 \times 10^0 + 2 \times 10^{-1} + 7 \times 10^{-2}$$

The Mechanics of any Base

From the above, the following mechanics for any base, say b , should be obvious:

1. there would be b unique digits in that base
2. the positional values, leftwards of the decimal place, would be b^0 , b^1 , b^2 , and so on. And the positional values rightwards of the decimal place would be b^{-1} , b^{-2} , b^{-3} and so on.

Converting a number in base, say b , to its decimal equivalent...

Thus, any number, say $\dots xyxw.pq\dots$ in base b would have a value equivalent to

$$(\dots xyzw.pq\dots)_b = \dots + x \times b^3 + y \times b^2 + z \times b + w + p \times b^{-1} + q \times b^{-2} + \dots$$

In this text, we would write a number in a base other than 10 in brackets with the base being specified as a sub-script. Thus 352 represents the number 352 of the decimal system, but $(352)_8$ represents the number 352 of base 8.

Thus,

$$(472)_8 = 4 \times 8^2 + 7 \times 8 + 2 = 314$$

$$(1101001)_2 = 1 \times 2^6 + 1 \times 2^5 + 1 \times 2^3 + 1 = 105$$

$$(243.41)_5 = 2 \times 5^2 + 4 \times 5 + 3 + \frac{4}{5} + \frac{1}{25} = 73 + 0.8 + 0.04 = 73.84$$

$$(253.14)_6 = 2 \times 6^2 + 5 \times 6 + 3 + \frac{1}{6} + \frac{4}{36} = 105 + 0.1666... + 0.111... = 105.2777...$$

The above also shows the method of converting a non-decimal number to its equivalent decimal value. The number obtained on the RHS is the decimal equivalent because when we do any calculation, say $4 \times 8^2 = 256$ or say $64 + 32 + 8 + 1 = 105$, we are following base 10. These multiplications or additions do not give the same results when considered in a base other than 10. Thus the value on the RHS is in decimal system.

To convert a number from decimal system to a non-decimal base, say b , ...

Divide the given number by the b and find the remainder and the quotient. Divide the quotient again by b and find the remainder and the quotient. Keep doing so till the quotient is 0. The remainders so found, in the reverse order (i.e. starting with the last remainder as the leading digit and the remainder from the first division as the unit digit) is the equivalent number in base b .

E.g. 1: Convert 314 to base 8.

8	<u>314</u>	<i>remainders</i>	
8	<u>39</u>		2
8	<u>4</u>		7
8	0		4

Thus the equivalent number in base 8 is $(472)_8$.

E.g. 2: Convert 105 to base 2.

2	<u>105</u>	<i>remainders</i>	
2	<u>52</u>		1
2	<u>26</u>		0
2	<u>13</u>		0
2	<u>6</u>		1
2	<u>3</u>		0
2	<u>1</u>		1
2	0		1

Thus the equivalent number in base 2 is, starting from reverse, $(1101001)_2$.

If the given number has a decimal part to it, multiply the decimal part by b and the integral part of the product will be the first number to the right of the decimal. Multiply the decimal part of the just found product again by b . The integral part of the new found product will be the second digit to the right of the decimal. The process is continued till the product found is an integer or ad-infinity.

E.g. 3: Convert 73.84 to base 5.

Converting the integral part 73 as learnt above,

$$\begin{array}{r|l} 5 & \underline{73} \quad \text{remainders} \\ & 3 \\ 5 & \underline{14} \\ & 4 \\ 5 & \underline{2} \\ & 0 \end{array}$$

Thus the integral part of the answer is 243. Now for the decimal part $0.84 \times 5 = 4.2$. Thus 4 is the first digit to the right of decimal.

$0.2 \times 5 = 1.0$. Thus the second digit to the right of decimal is 1 and these are the only two digits to the right of decimal.

Thus the equivalent number in base 5 is $(243.41)_5$

E.g. 4: Convert 105.2777... to base 6.

The integral part can be found as earlier to be 253. For the decimal part...

Since the decimal part 0.2777 is a recurring number, for exact calculations and also to lessen our work, we should first convert the recurring part to p/q form. As learnt in number systems, the p/q form of the recurring

$$\text{number } 0.27777 \text{ is } \frac{27-2}{90} = \frac{25}{90} = \frac{5}{18}.$$

Now, $0.27777 \times 6 = \frac{5}{18} \times 6 = 1\frac{2}{3}$. Thus the first digit to the right of the decimal is 1.

Next, $\frac{2}{3} \times 6 = 4.0$. Thus the second and the last digit to the right of decimal is 4.

Thus the required base-6 equivalent is $(253.14)_6$.

E.g. 5: Convert 80.08 to base 9.

$(80)_9 = 88$. For the decimal part...

$0.08 \times 9 = 0.72$. Thus the first digit to the right of the decimal is 0.

$0.72 \times 9 = 6.48$. The second digit in the decimal part is 6.

$0.48 \times 9 = 4.32$. The third digit in the decimal part is 4.

$0.32 \times 9 = 2.78$. The fourth digit in the decimal part is 2.

And so on, the process can continue to the required degree of precision.

Thus, $80.08 = (88.0642\dots)_9$

Base greater than 10

What about a base greater than 10? Say base 16 (hexa-decimal system). In this base we would need 16 unique digits. 0 to 9 are just 10 unique digits. We cannot use 10 to represent the tenth unique digit because then, say in the number 86104, we would not understand if the 10 stood for the tenth unique digit or for two digits 1 and 0. Hence we need a ‘unique’ digit. For this, we use A, B, C, ... to represent the 10, 11, 12, Thus 15 would be F. Rest of the working is exactly similar to the above. When converting a hexadecimal number to decimal, while doing the calculations, use 10, 11, ... in place of A, B, And while converting a decimal number to hexa-decimal, while dividing by 16, if the remainder is 10, 11, ..., use A, B, ... to represent the remainders in the hexa-decimal form.

We say the above is the mechanics of a base system, because this only helps us in mechanically solving questions. For a thorough understanding of base system, read the following section “Understanding How to Count in any Base”. Reading the section will give you a far better way to convert a non-decimal number to its decimal equivalent. Also it will help you add and subtract numbers in non-decimal base easily.

Understanding How to Count in any Base

It would be a good idea to see how only the 10 digits (0 to 9) can be used to depict millions and billions...

Consider we have to represent the number of marbles that one has (shown as circles in the figures below).

We have just 10 unique digits with us. Each of them can be used to mean/imply a specific number of marbles. And as we are accustomed to the decimal system, these digits are each used to represent the number of marbles shown besides each digit:

0		5	○ ○ ○ ○ ○
1	○	6	○ ○ ○ ○ ○ ○ ○
2	○ ○	7	○ ○ ○ ○ ○ ○ ○ ○
3	○ ○ ○	8	○ ○ ○ ○ ○ ○ ○ ○ ○
4	○ ○ ○ ○ ○	9	○ ○ ○ ○ ○ ○ ○ ○ ○ ○

The first problem arises when the number of marbles we have exceeds the cases shown above, say we have marbles numbering as follows:

○ ○ ○ ○ ○ ○ ○ ○ ○ ○

(Don't be in a haste to say we have 10 marbles, the symbol "10" has as yet not been defined. And also understand that "10" stands for as many marbles as above ONLY in the decimal system. That is precisely what this text is trying to imply, how to count with a given set of unique digits. A outcome of this learning should also be able to count when we have, say just 0, 1, 2, 3, 4 and 5 digits or say when we have 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E and F as the unique digits)

To represent these many marbles, we do not have any other unique digit left. And we can't use any of the already used one or else there will be confusion as to what number of marbles does the digit refer.

To overcome this problem, we make the use of two digits in conjunction: since one full cycle of using the unique digits is over, we prefix 1 and then start the cycle of using the unique digits once again (as the number written to the right of 1). Since this is the first case, after completion of the one cycle of using all the digits, this number of marbles will be represented as 10. Similarly, having one more marble would represent 11, having yet one more would represent 12 and so on.

10	○ ○ ○ ○ ○ ○ ○ ○ ○ ○	15	○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○
11	○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○	16	○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○
12	○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○	17	○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○
13	○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○	18	○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○
14	○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○	19	○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○

Next, when we have to represent as many marbles as following:

○ ○ ○ ○ ○ ○ ○ ○ ○ ○
○ ○ ○ ○ ○ ○ ○ ○ ○ ○

we again have run out of unique digits to fill in the question mark in "1?" (We have used all the unique digits in the right hand position and repeating any one would again lead to confusion). As you all must know, we overcome this problem by writing 2 as the left hand digit and repeat each of 0 to 9 as the right hand digit to represent marbles numbering from 20 to 29.

Thus, by now you would have got the logic of it all.

Next major hurdle arises after we represent the number of marbles, that we imagine when we say 99, by the number 99. If we have one more marble, we have also run out of options for the left hand digit. This is the time, we prefix one more position and thus having 1 more marble than 99 is represented by three digits: 100. Post this, we start writing the two right-most digits as 01, 02, ...09, 10, 11, ..., 19, 20, 21, ..., 99 to result in numbers 101, 102, ..., 199.

After this is a repetition of the above logic.

NOTE ABOUT NOTATION:

Since drawing many marbles is difficult, they will now onwards be represented by a number written in *italics* and in 'quotes'. While the number used is the decimal equivalent, since we are used to it and it immediately conjures up an image of the number of marbles, understand that we are using the notation to refer to *a number of marbles*. It is important to look at '15' as the number of marbles rather than the verbal - fifteen. Because the picture of marbles that '15' paints in the mind would remain the same number of marbles irrespective of the base, but when they are expressed in different base, the representation could be 15 (in decimal) or 17 (in octal) or 23 (in base '6') or even 1111 (in base '2').

Now consider we have to count millions and billions but we have only 8 unique digits viz 0 to 7. This is Octal system.

As seen above the first of the eight cases, can be easily assigned each of the eight unique digits...

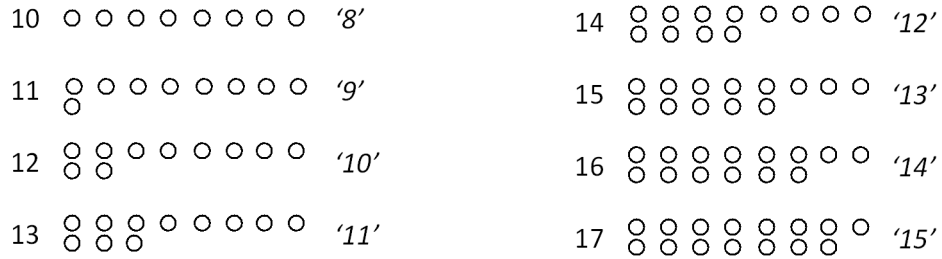
0		4	○ ○ ○ ○
1	○	5	○ ○ ○ ○ ○
2	○ ○	6	○ ○ ○ ○ ○ ○
3	○ ○ ○	7	○ ○ ○ ○ ○ ○ ○

Let's say we have one more marble, i.e. the number of marbles with us is

○ ○ ○ ○ ○ ○ ○ ○

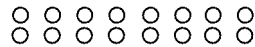
Please do not say these are 8 marbles. In the Octal system, there is nothing called 8. When we say these are 8 marbles, we refer to the notation 8 of the decimal system for representing these many number of marbles.

In the Octal system, we do not have any further unique digits and thus we start using two digits, starting with the left digit being 1. This being the first case after exhausting all the unique digits, the right digit can be the first unique digit i.e. 0. Thus, these many marbles would be represented as 10.



As seen above 16 may not mean '16' marbles when we use a non-decimal system. E.g. look at the last figure seen above, the number 16 represents as many marbles as we call '14'.

Next, when we have to represent as many marbles as following:



we again have run out of unique digits to fill in the question mark in "1?" (We have used all the unique digits in the right hand position and repeating any one would again lead to confusion). Now we increment the left digit to 2 and start using 0 onwards again. Thus these many marbles (16 in the decimal system) will be represented by 20 in the Octal system.

The process will now proceed in the same logic for larger numbers.

The figure below tries to capture how different number of marbles will be represented in different base. Try to follow a particular base vertically downwards and see how the counting happens when we have different number of unique digits...

Number of marbles	Base = 12	Base = 10	Base = 8	Base = 6	Base = 3	Base = 2
○	1	1	1	1	1	1
○ ○	2	2	2	2	2	10
○ ○ ○	3	3	3	3	10	11
○ ○ ○ ○	4	4	4	4	11	100
○ ○ ○ ○ ○	5	5	5	5	12	101
○ ○ ○ ○ ○ ○	6	6	6	10	20	110
○ ○ ○ ○ ○ ○ ○	7	7	7	11	21	111
○ ○ ○ ○ ○ ○ ○ ○	8	8	10	12	22	1000
'9'	9	9	11	13	100	1001
'10'	A	10	12	14	101	1010
'11'	B	11	13	15	102	1011
'12'	10	12	14	20	110	1100
'13'	11	13	15	21	111	1101
'14'	12	14	16	22	112	1110
'15'	13	15	17	23	120	1111
'16'	14	16	20	24	121	10000
'17'	15	17	21	25	122	10001
'18'	16	18	22	30	200	10010
'19'	17	19	23	31	201	10011
'20'	18	20	24	32	202	10100
'21'	19	21	25	33	210	10101
'22'	1A	22	26	34	211	10110
'23'	1B	23	27	35	212	10111
'24'	20	24	30	40	220	11000



Facets of counting in the decimal system that can be applied to other base as well:

In the Decimal System...

1. We have '10' unique digits, 0, 1, 2, ..., 9,

1. To represent '0' to '9', we need just single digit numbers 0, 1, 2, ..., 9.

2. We have to use two digit to represent '10' (worthwhile to notice that the first representation of two digits, 10 is needed to represent '10').

To represent '17', we first account for '10' marbles and then we are left with '7'. Since we had to account for just one lot of '10' the leading digit is 1 and since the remaining '7' is represented by 7, the representation is $1 \times 10 + 7 = 17$.

To represent '34', we have to account for three lots of '10' and left-over's of '4'. Thus the number is $3 \times 10 + 4 = 34$

3. We have to use three digits to represent '100' i.e. '10²' (again worthwhile to notice that the first representation of three digits, 100 is needed to represent '10²').

To represent '185', we have to account for one lot of '10²' and left-over's are '85'. Thus, the leading digit in the representation is 1. To represent '85', as seen earlier, we have to first account for eight lots of '10' and hence the next digit in the representation is 8. The left-over's now is '5' and is represented by the digit 5 as the third digit. (In standard representation, $1 \times 10^2 + 8 \times 10 + 5 = 185$)

To represent '703', we have to account for seven lots of '100' and hence the leading digit is 7. Now we are left with just 3 and hence there are no lots of '10' that need to be accounted. Hence the second digit is 0 and the unit's digit is 3. (In standard representation $7 \times 10^2 + 0 \times 10 + 3 = 703$)

4. We have to use four digits to represent '1000' i.e. '10³' i.e. the first representation of four digits, 1000 is needed to represent '10³'

Proceeding in the same manner for base 8...

1. We have '8' unique digits, 0, 1, 2, ..., 7,

1. To represent '0' to '7', we need just single digit numbers 0, 1, 2, ..., 7.

2. We have to use two digit to represent '8' (worthwhile to notice that the first representation of two digits, 10 is needed to represent '8').

To represent '13', we first account for '8' marbles and then we are left with '5'. Since we had to account for just one lot of '8' the leading digit is 1 and since the remaining '5' is represented by 5, the representation is $1 \times 8 + 5 = 13$.

To represent '34', we have to account for four lots of '8' and left-over's of '2'. Thus the number is represented as $4 \times 8 + 2 = 34$

3. We have to use three digits to represent '64' i.e. '8²' (again worthwhile to notice that the first representation of three digits, 100 is needed to represent '8²').

To represent '85', we have to account for one lot of '8²' and left-over's are '21'. Thus, the leading digit in the representation is 1. To represent '21', as seen earlier, we have to first account for two lots of '8' and hence the next digit in the representation is 2. The left-over's now is '5' and is represented by the digit 5 as the third digit. (In standard representation, $1 \times 8^2 + 2 \times 8 + 5 = 125$)

To represent '260', we have to account for four lots of '64' and hence the leading digit is 4. Now we are left with just 4 and hence there are no lots of '8' that need to be accounted. Hence the second digit is 0 and the unit's digit is 4. (In standard representation $4 \times 8^2 + 0 \times 8 + 4 = 404$)

4. We have to use four digits to represent '512' i.e. '8³' i.e. the first representation of four digits, 1000 is needed to represent '8³'

Now converting decimal numbers to non-decimal base will be easier. So taking up the same examples as done earlier...

E.g. 6: Convert 314 to base 8.

Since base is 8, we will try to write the number as multiples of 8, 64, 512...

Since the number is less than 512, we don't need a multiple of 512.

Finding a multiple of 64, that is as close to 314 but less than 314, we arrive at $4 \times 64 = 256$ and we are left over with 58. Obviously the left-over has to be less than 64 or else we would have taken a higher multiple of 64.

Now the left-over of 58 can be accounted by 7×8 and a leftover of 2. Thus,
 $314 = 4 \times 8^2 + 7 \times 8 + 2 = (472)_8$

E.g. 7: Convert 105 to base 2.

Since base needed is 2, try expressing 105 using powers of 2 i.e. using 1, 2, 4, 8, 16, 32, ...

We would need a 64 and leftover would be 41. Thus leading digit is 1.

To account for 41, we would need 32 (second digit is also 1) and now the leftover would be 9.

To account for 9, we do NOT need 16 (third digit is 0) but need an 8 (fourth digit is 1). Now the left-over is 1. To account for 1, we need neither a 4 nor a 2 (fifth and sixth digit is 0) and the leftover 1 will be the seventh digit. Thus the equivalent base 2 representation is 1101001. If the oral way is not clear, see the following working...

$105 = 1 \times 64 + 1 \times 32 + 0 \times 16 + 1 \times 8 + 0 \times 4 + 0 \times 2 + 1 = (1101001)_2$.

From now onwards we will convert a decimal number to a non-decimal base in this manner only and not the division process as learnt earlier. So study the following conversions to be accustom yourself to...

Converting from decimal to base 8, $(137)_{10} = 8^2 \times 2 + 8^1 \times 1 + 8^0 \times 1 = (211)_8$

Converting from decimal to base 5, $(276)_{10} = 5^3 \times 2 + 5^2 \times 1 + 5^1 \times 0 + 5^0 \times 1 = (2101)_5$

Converting from decimal to base 2, $(67)_{10} = 2^6 \times 1 + 2^5 \times 0 + 2^4 \times 0 + 2^3 \times 0 + 2^2 \times 0 + 2^1 \times 1 + 2^0 \times 1 = (1000011)_2$

Converting from decimal to base 12, $(3241)_{10} = 12^3 \times 1 + 12^2 \times A + 12^1 \times 6 + 12^0 \times 1 = (1A61)_{12}$

(Knowing that $12^3 = 1728$ would help you vastly in the above calculation)

Also the above method helps us to quickly do mental working in converting smaller decimal numbers to any required base...

13 when converted to base 8 will be $(15)_8$ because '13' is 5 more than one multiple of 8

20 when converted to base 8 will be $(24)_8$ because '20' is 4 more than the second multiple of 8.

14 when converted to base 9 will be $(15)_9$ because it is 5 more than 9.

20 when converted to base 3 has to be a three digit number since we would need to use 3^2 . Since two lots of 3^2 accounts for 18 and there is a leftover of only 2, the equivalent number in base 3 is $(202)_3$.

47 when converted to base 5 has to $25 + 4 \times 5 + 2$ and hence is $(142)_5$.

These small calculations will help us in adding and subtracting...

Adding, subtracting or multiplying in non-decimal base:

Consider adding $87 + 98$ in the decimal system...

We first add 7 and 8 and since the answer, 15, is 5 more than 1×10 , ten being the base, we carry forward 1 and write 5 as the unit's digit of the answer.

Similarly, when we add $8 + 9 + 1$, since the answer, 19, is 9 more than 10, we carry forward 1 and write 9 as the ten's digit. Thus the answer is 195.

Thus, whenever the addition of a column results in a number greater than the base, we carry forward whatever is in excess of the base and write the remaining as the addition of that column..,

E.g. 8: Let's try to add two numbers $(526)_8$ and $(477)_8$.

First we add 6 and 7 and since the answer, 13, is 5 more than the base, we carry forward 1 and write 5 as the unit's digit.

Next we add, $2 + 7 + 1$ and since the answer, 10, is 2 more than the base, we carry forward 1 and write 2 as the ten's digit.

Proceeding in a similar manner we get the answer to be 1225

$$\begin{array}{r} 1 \quad 1 \quad 1 \\ 5 \quad 2 \quad 6 \\ + 4 \quad 7 \quad 7 \\ \hline 1 \quad 2 \quad 2 \quad 5 \end{array}$$

To understand why we carry forward 1 and not 8, look at the following working...

Step I: Starting the addition from the rightmost digits, $6 + 7 = 13$ but that is in decimal system. In base 8, it is equivalent to $(15)_8$. Hence we write 5 in the last row, and carry the 1.

Step II: Next step is $1 + 2 + 7 = (10)_{10} = (12)_8$. Again 1 is carried over.

Step III: Next, $1 + 5 + 4 = (10)_{10} = (12)_8$. Hence we get the sum as $(1225)_8$.

E.g. 9: Find the sum of $(267)_9$, $(867)_9$ and $(127)_9$.

Adding $7 + 7 + 7$, since the answer, 21, is 3 more than the twice the base, we write the unit's digit as 3 and the carry-over is 2. Apart from this change the rest is the same as above

$$\begin{array}{r} 1 \quad 1 \quad 2 \\ 2 \quad 6 \quad 7 \\ + 8 \quad 6 \quad 7 \\ + 1 \quad 2 \quad 7 \\ \hline 1 \quad 3 \quad 7 \quad 3 \end{array}$$

Step I: Adding $7 + 7 + 7$, we get $(21)_{10} = (23)_9$. 3 is written and 2 is carried over.

Step II: Adding $2 + 6 + 6 + 2$, we get $(16)_{10} = (17)_9$. 7 is written and 1 is carried over.

Step III: $1 + 2 + 8 + 1$ gives $(12)_{10} = (13)_9$. 3 is written and 1 is carried over.

In subtraction, when we have to subtract a larger number from a smaller number, we borrow 1 from the digit to the left. In decimal system, this borrowing of 1 from the left position is equivalent to borrowing 10 units. In any other base, say b , borrowing 1 unit from the position on the left is equivalent to borrowing b .

E.g. 10: Find the difference between $(4076)_9$ and $(3777)_9$.

Since 7 has to be subtracted from 6, we borrow 1 (i.e. 9 since 1 of ten's digit is same as 9 in unit's position) from the left hand digit. Thus, for the unit's place the calculation becomes $6 + 9 - 7 = 8$.

Now the right hand digit becomes 6 since 1 has been lent to the unit's place. Now we have to subtract 7 from 6. Thus, we have to again borrow 1 from the hundred's position. However the hundred's position is 0 and we cannot borrow any digit from it. Thus first we have to borrow 1 from the thousand's position and this borrowed 1 becomes 9 when considered at the hundred's place. Now borrowing 1 from the hundred's place for the ten's place, the answer for the ten's place is $6 + 9 - 7 = 8$.

The result in the hundred's place will be $8 - 7 = 1$ and the result in the thousand's place will be $3 - 3 = 0$. Thus answer is $(188)_9$

$$\begin{array}{r}
 \overset{3}{\cancel{4}} \quad \overset{8}{\cancel{0}} \quad \overset{9}{\cancel{7}} \quad \overset{9}{\cancel{6}} \\
 - \quad 3 \quad 7 \quad 7 \quad 7 \\
 \hline
 \quad \quad 1 \quad 8 \quad 8
 \end{array}$$

E.g. 11: Compute $(52)_6 - (15)_6 - (24)_6$

Since we have to subtract $2 - 5 - 4$ even if we borrow 1 unit from the ten's position, we would have to do $(2 + 6) - 5 - 4$ which would still be negative. Thus we would have to borrow two unit's from the ten's position and this is equivalent to having an additional 12 in the unit's place. Thus the answer to the unit's place would be $(2 + 12) - 5 - 4 = 5$. And the answer at the ten's place would be $(5 - 2) - 1 - 2 = 0$. Thus the answer would be 5.

E.g. 12: Find $(356)_8 \times (741)_8$.

$$\begin{array}{r}
 \quad \quad \quad 3 \quad 5 \quad 6 \\
 \quad \quad \times \quad 7 \quad 4 \quad 1 \\
 \hline
 \quad \quad \quad 3 \quad 5 \quad 6 \\
 1 \quad 6 \quad 27 \quad 30 \quad \times \\
 3 \quad 2 \quad 50 \quad 52 \quad \times \quad \times \\
 \hline
 3 \quad 3 \quad 7 \quad 14 \quad 5 \quad 6
 \end{array}$$

Step I: $1 \times 6 = 6$, $1 \times 5 = 5$, $1 \times 3 = 3$

Step II: $4 \times 6 = (24)_{10} = (30)_8$, so 0 is written and 3 is carried over.

$4 \times 5 + 3 = (23)_{10} = (27)_8$. So 7 is written and 2 is carried over.

$4 \times 3 + 2 = (14)_{10} = (16)_8$.

Step III: $7 \times 6 = (42)_{10} = (52)_8$, $7 \times 5 + 5 = (40)_{10} = (50)_8$,

$7 \times 3 + 5 = (26)_{10} = (32)_8$.

In case of questions in which the numbers are in two different bases, convert one of the numbers in the other base and then proceed.

E.g. 13: Find $(246)_8 \times (654)_7$. Express the answer in base 8.

$$(654)_7 = 7^2 \times 6 + 7^1 \times 5 + 7^0 \times 4 = (333)_{10}$$

$$(333)_{10} = 8^2 \times 5 + 8^1 \times 1 + 8^0 \times 5 = (515)_8$$

and now, for the multiplication,

$$\begin{array}{r} 2 4 6 \\ \times 5 1 5 \\ \hline 1 4 27 36 \\ 2 4 6 \times \\ 1 4 27 36 \times \times \\ \hline 1 5 13 17 15 6 \end{array}$$

Thus, the answer is $(153756)_8$

Converting from one non-decimal base to another non-decimal base:

To convert from one base to another, the easiest method is to convert first to decimal base and then to the required base.

E.g. 14: Convert from base 6 to base 8:

$$(128)_6 = 6^2 \times 1 + 6^1 \times 2 + 6^0 \times 8 = 36 + 12 + 8 = (56)_{10}$$

$$(56)_{10} = 8^1 \times 7 + 8^0 \times 0 = (70)_8$$

E.g. 15: Convert from base 12 to base 9:

$$(50AB)_{12} = 12^3 \times 5 + 12^2 \times 0 + 12^1 \times 10 + 12^0 \times 11 = (8771)_{10}$$

$$(8771)_{10} = 9^4 \times 1 + 9^3 \times 3 + 9^2 \times 1 + 9^1 \times 4 + 9^0 \times 6 = (13146)_9$$

Note: We can also convert directly from one base to another without converting it to decimal first but for that a lot of calculations need to be done in a base other than decimal, which most of us are not very comfortable with. Hence, we suggest you to follow the above given method only.

Practice

The entrance exams do not have questions like the following. They are given only for you to get comfortable with converting numbers in different bases and to perform basic operations like addition, multiplication in bases other than decimal. No options are provided for these questions.

- Convert the following numbers to decimal system:
 - 465 in base 7
 - 1289A in base 11
 - 5216 in base 8
 - 7614.25 in base 9
- The following numbers are given in decimal system. Convert them to the specified bases.
 - 362 to base 4
 - 1892 to base 7
 - 2198 to base 11
 - 901.42 to base 4
- Convert the following from the base in which they are given to the base specified:
 - 161 in base 7 to base 8
 - 101101010 from base 2 to base 8
 - 27AB5D from base 16 to base 4
 - 20112.012 from base 3 to base 9
- Add the following.
 - $(1763)_8$ and $(14526)_8$. Express your answer in base 8 itself.
 - $(5A0B)_{12}$ and $(69B4)_{12}$. Express your answer in base 12 itself.

Answers are given at the end of the exercise on this topic, on next page.

There have been only 3 questions on base system through all CAT papers from 1999 to 2007. While one of the questions was a straightforward conversion from base 10 to base 12, the others were a little more innovative. Thus the questions would not be of the type of conversions as given above. They were given so that one gets a basic idea of working in various bases. Look at the following examples that show that almost every question of decimal system can also be asked in any other base. And that the types of questions are not just limited to conversion between bases.

E.g. 16: If in a particular base, $15^2 = 251$, find the square of 25 in that base.

Method 1:

If the base used is b , then $(b + 5)^2 = 2b^2 + 5b + 1$.

Solving for b , we get, $b^2 + 10b + 25 = 2b^2 + 5b + 1$ i.e. $b^2 - 5b - 24 = 0$. Thus $b = 8$ or -5 . Since the base cannot be negative, the base is 8.

Thus, $(25)_8 \times (25)_8 = (21^2)_{10} = (441)_{10} = 6 \times 64 + 7 \times 8 + 1 = (671)_8$

E.g. 18: In base 11, when the digits of a certain two-digit number are reversed, we get a value that is thrice the value of the original number. Find the number.

Method 1: Consider the number to be $(xy)_{11}$. When the digits are reversed we get the number $(yx)_{11}$. Now by the relation given $(yx)_{11} = 3 \times (xy)_{11}$

Converting the numbers in decimal to save mental work and solving the above, $11y + x = 3 \times (11x + y) \Rightarrow 11y + x = 33x + 3y \Rightarrow 8y = 32x \Rightarrow y = 4x$

Now, if $x = 1$, then $y = 4$ and if $x = 2$, $y = 8$ (x cannot take values greater than 2 or else y would not be a single digit). Thus there are two such possible numbers i.e. $(14)_{11}$ and $(28)_{11}$.

Make sure you check that $(41)_{11}$ is thrice of $(14)_{11}$ and also that $(82)_{11}$ is thrice of $(28)_{11}$. This would lend you more confidence in the above working.

Method 2:

$$\begin{array}{r} x \ y \\ \times \quad 3 \\ \hline y \ x \end{array}$$

Case 1: $3 \times y$ is less than the base 11 i.e. there is no carry-over. In this case we would have $3y = x$ and $3x = y$, which is not possible.

Case 2: $3 \times y$ is 11 or more but less than 22 i.e. there is a carry-over of 1. In this case we would have $3y - 11 = x$ and $3x + 1 = y$. Substituting value of x from first relation in the second relation, we have $3(3y - 11) + 1 = y$ i.e. $y = 4$ and $x = 1$. This gives the first solution $(14)_{11}$.

Case 3: $3 \times y$ is 22 or greater than 22 but less than 33 i.e. there is a carry-over of 2. In this case we would have $3y - 22 = x$ and $3x + 2 = y$. Substituting value of y from second relation in the first relation, we have $3(3x + 2) - 22 = x$ i.e. $x = 2$ and $y = 8$. This gives the second solution $(28)_{11}$.

These can be the only possible cases because y is a digit in base 11 and hence $3 \times y$ cannot yield a carry-over of 3.

Exercise:

- In a certain base $11 \times 13 = 99$. Find the base
 (1) 4 (2) 5 (3) 6 (4) Data Inconsistent
- How many 6 digit numbers exists in base 6. Express your answer in base 6 and also in decimal base, in order.
 (1) 900,000; 600,000 (2) 500,000; 5×6^5 (3) 600,000, 6^6 (4) 555,555; 5×6^5
- A two digit decimal number when expressed in base 2 has it's unit's digit as 1, when expressed in base 3 has it's unit digit 1 and when expressed in base 5 also has it's unit digit 1. How many such two-digit numbers exists?
 (1) 3 (2) 18 (3) 30 (4) 45
- What is the sum of all 2 digit numbers of base 8? Express your answer in base 8 itself.
 (1) 2436 (2) 4604 (3) 4064 (4) 2076
- In base 12, the number, 359A3Bx, where x is a digit, is divisible by B. What is the value of x?
 (1) 1 (2) 2 (3) 3 (4) B

Answers to practice questions:

- a. 243 b. 1671 c. 2702 d. 5602.28
- a. 11222 b. 5342 c. 1719 d. 32011.12
- a. 134 b. 552 c. 130222231131 d. 315.16
- a. 16511 b. 10703

Answer Key

Set Theory

1. 2 2. 3 3. 1 4. 2 5. 4
6. 1 7. 1 8. 1 9. 3 10. 2
11. 1 12. 3 13.a. 4 13.b. 3 13.c. 1
13.d. 2 14. 1 15. 3 16. 4 17. 1
18. 3

Coordinate Geometry

1. 4 2. 1 3. 1 4. 3 5. 1
6. 3 7. 2 8. 4 9. 2 10. 2

Heights & Distances

1. 3 2. 2 3. 4 4. 1 5. 4

Calenders

1. 1 2. 3 3. 1 4. 2 5. 4

Base Systems

1. 4 2. 2 3. 1 4. 2 5. 3



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